Multenions and Differential Invariants.—II.

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(Communicated by W. B. Hardy, Sec.R.S. Received June 22, 1921.)

§ 11. *The fundamental covariant vector linity*, η. The meaning of a manifold as based on a quadratic differential form is here taken for granted, and this, whether, in the ordinary terminology, all or only some of its dimensions are real. Let

$$\left.\begin{aligned} \iota_a &= i_a, \qquad (a = 1,\ 2,\ \ldots c) \\ \iota_b &= i_b\sqrt{(-1)}, \qquad (b = c+1,\ c+2,\ \ldots n) \\ \rho &= \sum_{h=1}^{n} x_h \iota_h \end{aligned}\right\}, \tag{1}$$

c having any given value from 1 to n. In the present paper we shall deal with multenions and multenion linities based on a system of vectors ρ for which all the co-ordinates x_h and the similar scalars are real. In the case of relativity $n = 4$, $c = 3$.

The fundamental quadratic form is taken to be

$$ds^2 = \mathrm{V}_0 d\rho\eta d\rho = \mathrm{V}_0 d\rho^{\backprime}\eta^{\backprime} d\rho^{\backprime}, \tag{2}$$

the sign which is given to ds^2, a scalar, being convenient in applications to relativity, but inconvenient in applications to pure geometry. η, a self-conjugate vector linity, is a given function of position, ρ; $\rho^{\backprime}$, a vector, an arbitrary function of ρ; and since the transformation from η to $\eta^{\backprime}$ is given by (2) the fundamental linity η is said to be covariant. [η and ds^2 are real, but sometimes ds and sometimes $ds\sqrt{(-1)}$ is real; when ds is real it is "a time-like interval"; when $ds\sqrt{(-1)}$ is real it is a length; when the question of reality is left undecided ds is an interval.] At any given position $\rho^{\backprime}$ may be so chosen that $\eta^{\backprime} d\rho^{\backprime} = d\rho^{\backprime}$, that is $\eta^{\backprime} = 1$. This imposes that η is without singularities; it has n mutually orthogonal principal directions and all its roots are real and positive, excluding zero values. Thus $\eta^{\frac{1}{2}}$ is assumed to have the same orthogonal system and its roots are taken to be the positive values of the square roots of η so that $\eta^{\frac{1}{2}}$ is real and without ambiguity.

We may, and from here on shall, take our previously introduced scalar k (§ 8) to be $|\eta^{\frac{1}{2}}|$ the determinant of $\eta^{\frac{1}{2}}$. [We have

$$\eta = \chi' \eta^{\backprime} \chi$$

so that

$$|\eta| = |\eta^{\backprime}| \times |\chi|^2 = |\eta^{\backprime}| \times (k/k^{\backprime})^2,$$

because $k/k^{\backprime}$ was defined by

$$k db = k^{\backprime} db^{\backprime} \quad \text{or} \quad k/k^{\backprime} = |\chi|.]$$

It will be seen that when, in the usual notation of relativity, it is said that with Galilean co-ordinates

$$(g_{11}, g_{22}, g_{33}, g_{44}) = (-1, -1, -1, +1),$$

we have not that $\eta(\iota_1, \iota_2, \iota_3, \iota_4) = (-\iota_1, -\iota_2, -\iota_3, +\iota_4)$ but instead the simpler statement that $\eta = \eta^{\frac{1}{2}} = 1$ or

$$\eta\,(\iota_1, \iota_2, \iota_3, \iota_4) = (\iota_1, \iota_2, \iota_3, \iota_4),$$

for it will be observed that this gives

$$ds^2 = -dx_1{}^2 - dx_2{}^2 - dx_3{}^2 + dx_4{}^2.$$

A little care is required in expressing a linity, its conjugate, reciprocal, etc., by scalars or co-ordinates. It seems sufficient to illustrate from vector linities. When we say that ϕ is given by the square array of 16 scalars

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ l & m & n & p \\ q & r & s & t \end{vmatrix}$$

we imply that the co-ordinates of $\phi\iota_1$ are found in the first *column* thus

$$\phi\iota_1 = a\iota_1 + e\iota_2 + l\iota_3 + q\iota_4$$

and so for the other columns. In the square array of n^2 scalars specifying ϕ, let a_{xy} be the scalar in the *xth row and yth column.* Thus $\phi\iota_y = \sum_{x=1}^{n} a_{xy}\iota_x$, or more conveniently

$$a_{xy} = \mathrm{V}_0\iota_x{}^{-1}\phi\iota_y. \tag{3}$$

Except when $c = n$ the square array of a self-conjugate linity is not symmetrical about the principal diagonal. Let a_{xy}, a_{xy}', be the square arrays of ϕ and its conjugate ϕ'. By the meaning of conjugate we have

$$\mathrm{V}_0\,\iota_x\phi\iota_y = \mathrm{V}_0\,\iota_y\phi'\,\iota_x.$$

Thus instead of $a_{yx}' = a_{xy}$ (as we have when $c = n$) we have by (3)

$$a_{yx}' = \iota_x{}^2\iota_y{}^2 a_{xy}, \tag{4}$$

and in particular, when ϕ is self-conjugate $a_{yx} = \iota_x{}^2\iota_y{}^2 a_{xy}$.

If we would retain the 16 scalars g_{ab} with their usual meanings as in treatises on relativity we may use $g_{(ab)}$ for the corresponding scalars specifying η. Thus from the fact that $ds^2 = \Sigma g_{ab}dx_a dx_b$ and from (3)

$$g_{ab} = \mathrm{V}_0\,\iota_a\eta\iota_b, \qquad g_{(ab)} = \mathrm{V}_0\,\iota_a{}^{-1}\eta\iota_b,$$

whence

$$g_{ab} = \iota_a{}^2 g_{(ab)}. \tag{5}$$

More details with regard to these notations and the theory of tensors will be found below in § 17.

Note on Notations in the Present Paper.—There will be many exceptions below to the conventions here to be described. Especially do exceptions occur in very frequently recurring symbols. Thus none of the following receive the "cross" notation although they should do according to the conventions,

$$\lambda,\ \mathrm{E}_\alpha,\ \bar{\mathrm{E}}_\alpha,\ \mathrm{C}_\omega,\ \mathrm{C},$$

λ is a covariant vector, and of the other four C_ω, C are contramixt linities while E_α, $\bar{\mathrm{E}}_\alpha$ are portions of contramixt linities. The explanations of the technical terms will be made as they come to be used below.

(1) Contravariant multenions will have no distinguishing mark. Covariant multenions will usually be distinguished by a "cross" (which, because of its position, is scarcely likely to be confused with a sign of multiplication), thus

$$\alpha^\times,\ u^\times,\ p^\times,\ q^\times,$$

read "alpha cross," etc.

(2) Covariant and comixt linities will have no distinguishing mark. Contravariant and contramixt linities will usually be distinguished by a cross.

(3) Contravariant and covariant linities will be denoted by Greek letters; contramixt and comixt linities by Roman capitals.

(4) $\alpha, \beta, \gamma, \alpha^\times, \beta^\times, \gamma^\times$, will generally denote "dummy" vectors. (Eddington uses "dummy" in an analogous sense for a positive integer.) Dummies are always (below, though not of necessity) treated as constants in that they are not affected by the differentiations of ∇.

The duty of a dummy contravariant or covariant multenion may be generally described as "to occupy, and thereby indicate, a situation into which an actual, previously defined, contravariant or covariant multenion, may legitimately be inserted."

§ 12. *The Invariant Forms of any Multenion Expression.*—η has been defined as a fundamental *vector* linity. It is also used for the corresponding extensive linity, a multenion linity; and similarly for $\eta^{\frac{1}{2}}$. Since in (2), of § 11, $ds^2 = (\eta^{\frac{1}{2}}d\rho)^2$, $\eta^{\frac{1}{2}}d\rho$ in our manifold takes the place of the $d\rho$ of multenions in a Galilean manifold. [In a Euclidean manifold all the dimensions are real. In a Galilean manifold some are imaginary.] More generally, at a given position of the manifold, $\eta^{\frac{1}{2}}q$ where q is a contravariant multenion, or $\eta^{-\frac{1}{2}}q^\times$ where $q^\times$ is a covariant multenion, takes the place of q in a Galilean manifold. $\eta^{\frac{1}{2}}q$ and $\eta^{-\frac{1}{2}}q^\times$ will be spoken of as the neutral forms of q and $q^\times$. Indeed, the three q (contravariant), ηq (covariant) and $\eta^{\frac{1}{2}}q$ (neutral), are to be regarded as only different mathematical representations of the *same* intrinsic property, of the manifold; just as in an ordinary dynamical system, the whole system of velocities or instead the whole system of momenta equally will serve to specify the instantaneous motion.

Corresponding to any multenion expression such as $V_a \,.\, pqV_b(rs)$ which has a Galilean interpretation there are always the three forms (p, q, r, s now being contravariant) in our manifold (contra., co., neut.).

$$\eta^{-\frac{1}{2}}V_a \,.\, [\eta^{\frac{1}{2}}p\eta^{\frac{1}{2}}qV_b(\eta^{\frac{1}{2}}r\eta^{\frac{1}{2}}s)], \qquad \eta^{\frac{1}{2}}V_a \,.\, [\ ], \text{ and } V_a \,.\, [\ ].$$

The first and second of these three can *always* be expressed in such a shape that only η and not $\eta^{\frac{1}{2}}$ occurs explicitly; though the expression may be very complicated as in the example chosen. It is these final forms from which $\eta^{\frac{1}{2}}$ has been explicity banished, that are regarded as invariantive.

To show this, first note as an example, that the product pqr can be first modified by expressing qr as a sum of terms such as $V_{a+b-2c} \,.\, V_aqV_br$ and then $p \,.\, qr$ can be similarly modified. We have then merely to find the invariant forms (two, namely, covariant and contravariant) of $V_{a+b-2c}uv$. [The reader is reminded that in our notation u, v, w, have homogeneities a, b, c, respectively, or have the forms

$$u = V_ap,\ v = V_bq,\ w = V_ar.]$$

Let a be greater or equal to b and let u, v be contravariant. Then most frequently all we require to note are the two cases $c = 0$, $c = b$.

$$\eta^{-\frac{1}{2}}V_{a+b}\eta^{\frac{1}{2}}u\eta^{\frac{1}{2}}v = V_{a+b}uv, \qquad \eta^{\frac{1}{2}}V_{a+b}\eta^{\frac{1}{2}}u\eta^{\frac{1}{2}}v = V_{a+b}\eta u\eta v,$$

$$\eta^{-\frac{1}{2}}V_{a-b}\eta^{\frac{1}{2}}u\eta^{\frac{1}{2}}v = V_{a-b}u\eta v, \qquad \eta^{\frac{1}{2}}V_{a-b}\eta^{\frac{1}{2}}u\eta^{\frac{1}{2}}v = V_{a-b}\eta uv,$$

and here of course we may have $u = \eta^{-1}u^\times$ or $v = \eta^{-1}v^\times$ in each of the forms.

To deal with the general value of c in $V_{a+b-2c}uv$ we use ζ. Now we know that if τ_1, τ_2, ... τ_n are n independent contravariant vectors at a point, then their normal reciprocals $\bar{\tau}_1$, $\bar{\tau}_2$, ... $\bar{\tau}_n$ are n independent covariant vectors. Also we know that if $(\alpha, \beta^\times)$ is any expression—bilinear in two vectors

$$(\zeta, \zeta) = -\sum_{a=1}^{n}(\tau_a, \bar{\tau}_a).$$

Hence if we insert ζ, ζ into two places in an invariant form, of which one is occupied by a contravariant vector and the other by a covariant vector, the resulting expression will be invariantive (meaning, "of invariant form"). Now it is easy to see by reduction to primitive units that

$$V_{a+b-2c}uv = [(-)^{\frac{1}{2}c(c+1)}/c\,!]\,V_{a+b-2c} \,.\, V_{a-c}u\zeta_c^{(c)}V_{b-c}\zeta_c^{(c)}v, \tag{1}$$

where as usual $\zeta^{(c)}$ stands for $\zeta_1\zeta_2\ldots\zeta_c$ and $\zeta_c^{(c)}$ for $V_c\zeta^{(c)}$.

Hence the contravariant form of $V_{a+b-2c}uv$ is

$$[(-)^{\frac{1}{2}c(c+1)}/c\,!]\,V_{a+b-2c} \,.\, V_{a-c}u\zeta_c^{(c)}V_{b-c}\eta\zeta_c^{(c)}v, \tag{2}$$

from which the covariant form may be written down, and either $u^\times = \eta u$ or $v^\times = \eta v$ or both may be used in place of u and v. In my MS. work I am in the habit of using $\zeta_{K}^{(c)}$, $\zeta_{Q}^{(c)}$ for $K\zeta_c^{(c)}$, $Q\zeta_c^{(c)}$. Thus (2) would be $(c\,!)^{-1}V_{a+b-2c}\,.\,V_{a-c}u\zeta_c^{(c)}V_{b-c}\eta\zeta_{K}^{(c)}v$. A special case of (2) often required is that the contravariant form of $V_2\omega\omega'$ where ω and ω' are contravariant and ${}_nV_2$ is

$$V_2(V_1\zeta\omega)(V_1\eta\zeta\omega'), \tag{3}$$

or more generally, for infinitesimal rotations generally, the contravariant form of $V_a\omega u$ is

$$V_a(V_1\zeta\omega)(V_{a-1}\eta\zeta u). \tag{4}$$

§ 13. *Normal and Incident Components in Invariant Form.*—These components occur very frequently in relativity. Thus the energy-momentum linity is density multiplied by an incident component, at any rate in its kinetic or primitive form. Normal and incident components serve to separate: (1) vector potential from electrostatic potential; (2) momentum from energy; (3) force from power; (4) current density from charge density; (5) magnetic induction from electrostatic force; (6) magnetic field strength from displacement. One should always be prepared to recognise these resolutions.

In quaternions if α is a given vector and σ an arbitrary one, σ has two components with reference to α, namely the normal component

$$N_0\sigma = \alpha^{-1}V\alpha\sigma = \alpha V\alpha\sigma\,.\,/\alpha^2$$

and the incident component

$$I_0\sigma = \alpha^{-1}S\alpha\sigma = \alpha S\alpha\sigma\,.\,/\alpha^2.$$

As will be familiar to users of the quaternion method, it is these components each multiplied by α^2, rather than the components themselves which we shall expect to meet with frequently in analytical expressions. The two

$$N_a\sigma = N\sigma = \alpha V\alpha\sigma, \qquad I_a\sigma = I\sigma = \alpha S\alpha\sigma,$$

may be referred to as the normal and incident *expressions* of σ with reference to α. With regard to these four linities N_0, I_0, N, I we have as follows

$$\left.\begin{array}{lll} N_0^2 = N_0, & I_0^2 = I_0, & N_0I_0 = 0 = I_0N_0 \\ N_0' = N_0, & I_0' = I_0, & N_0+I_0 = 1 = N_0'+I_0' \end{array}\right\}, \tag{1}$$

$$\left.\begin{array}{lll} N^2 = \alpha^2N, & I^2 = \alpha^2I, & NI = 0 = IN \\ N' = N, & I' = I, & N+I = \alpha^2 = N'+I' \end{array}\right\}. \tag{2}$$

Similarly in a Galilean manifold any multenion q has two components N_0q, I_0q normal and incident to a given vector α. Let

$$q = \sum_{c=1}^{n} w, \quad \text{where} \quad w = V_cq;$$

then when the linities Nw, N_0w, etc., have been defined, Nq, N_0q have also been defined. We have

(Galilean manifold.)

$$\left.\begin{array}{ll} Nw = \alpha V_{c+1}\alpha w, & Iw = \alpha V_{c-1}\alpha w \\ N_0w = \alpha^{-1}V_{c+1}\alpha w, & I_0w = \alpha^{-1}V_{c-1}\alpha w \end{array}\right\}. \tag{3}$$

[If $\acute{\epsilon}$ is a vectorium of homogeneity b, a vector σ has a normal component $\acute{\epsilon}^{-1}V_{b+1}\acute{\epsilon}\sigma$ and an incident component $\acute{\epsilon}^{-1}V_{b-1}\acute{\epsilon}\sigma$ with reference to $\acute{\epsilon}$. With reference to $\acute{\epsilon}$, w has many, not merely two, components given by

$$w = \acute{\epsilon}^{-1}V_{b+c}\acute{\epsilon}w + \acute{\epsilon}^{-1}V_{b+c-2}\acute{\epsilon}w + \ldots.$$

The terms on the right of this equation are all, separately, but not obviously, of homogeneity c.]

Next let α be a given contravariant vector at a given position ρ of our manifold and q an arbitrary contravariant multenion at the same position and let $w = V_cq$. Also let α_ν, q_ν, w_ν be the neutral forms and $\alpha^\times$, $q^\times$, $w^\times$ the covariant forms of α, q, w respectively, that is to say,

$$\left.\begin{array}{l} \eta^{\frac{1}{2}}\alpha = \alpha_\nu = \eta^{-\frac{1}{2}}\alpha^\times \\ \eta^{\frac{1}{2}}w = w_\nu = \eta^{-\frac{1}{2}}w^\times \\ \eta^{\frac{1}{2}}q = q_\nu = \eta^{-\frac{1}{2}}q^\times \end{array}\right\}. \tag{4}$$

The invariant form of α^2 is $\alpha_\nu{}^2$ or

$$V_0\alpha\eta\alpha = \alpha_\nu{}^2 = V_0\alpha^\times\eta^{-1}\alpha^\times. \tag{5}$$

With our present meanings (3) is not invariant in form. There are four modes of choosing the invariant forms of N and I, N_0 and I_0 and generally of any multenion linities, (1) covariant, (2) contravariant, (3) comixt, (4) contramixt. We will take number (3) of these for the basis of the notation in connection with N and I. A comixt linity means one which acting on a covariant multenion produces a covariant multenion. For this type of linity it is obvious that the invariant forms of (3) are given by

$$\left.\begin{array}{l} Nw^\times = V_c\alpha V_{c+1}\eta\alpha w^\times = V_c\eta^{-1}\alpha^\times V_{c+1}\alpha^\times w^\times \\ Iw^\times = V_c\eta\alpha V_{c-1}\alpha w^\times = V_c\alpha^\times V_{c-1}\eta^{-1}\alpha^\times w^\times \\ N_0w^\times = Nw^\times/V_0\alpha\eta\alpha = Nw^\times/V_0\alpha^\times\eta^{-1}\alpha^\times \\ I_0w^\times = Iw^\times/V_0\alpha\eta\alpha = Iw^\times/V_0\alpha^\times\eta^{-1}\alpha^\times \end{array}\right\}. \tag{6}$$

(2) comes with this notation to be modified in two ways only; α^2 is replaced by $V_0\alpha\eta\alpha$ or $V_0\alpha^\times\eta^{-1}\alpha^\times$ and in place of N' and I' we have a new kind of conjugate $\eta N'\eta^{-1}$, $\eta I'\eta^{-1}$ involving the fundamental linity η. Thus we now have

$$\left.\begin{array}{lll} N^2 = V_0\alpha\eta\alpha \, . \, N, & I^2 = V_0\alpha\eta\alpha \, . \, I, & NI = 0 = IN \\ \eta N'\eta^{-1} = N, & \eta I'\eta^{-1} = I, & N+I = V_0\alpha\eta\alpha = \eta(N'+I')\eta^{-1} \end{array}\right\}. \tag{7}$$

[Note that if we put $\eta\phi'\eta^{-1} = \phi_c$ we obtain for this new conjugate the familiar properties

$$(\phi+\psi)_c = \phi_c+\psi_c, \qquad (\phi\psi)_c = \psi_c\phi_c, \qquad (\phi_c)_c = \phi.]$$

The types of linity we have to recognise are given in the following short list.

Linity Types.

Name	Co-variant.	Contra-variant.	Co-mixt.	Contra-mixt.	Neutral.
Symbol	ξ	$\xi^\times$	X	$\mathrm{X}^\times$	$\mathrm{X}_\nu = \phi$
Meaning	$p^\times = \xi q$	$p = \xi^\times q^\times$	$p^\times = \mathrm{X} q^\times$	$p = \mathrm{X}^\times q$	$p_\nu = \mathrm{X}_\nu q_\nu$

Thus ξ converts a q (contravariant) into a $p^\times$ (covariant) and similarly for the other types.

The two mixed linities convert a multenion of a given type to one of the same type; the other two convert a given type to the *other* type. [In our present method the mixed linities could scarcely be more unhappily named.]

Corresponding to any one of the types we have, of course, by means of η, symbols of the other types with the same intrinsic significance. Thus, corresponding to the neutral form ϕ we have

$$\left.\begin{array}{ll} \xi = \eta^{\frac{1}{2}}\phi\eta^{\frac{1}{2}}, & \xi^\times = \eta^{-\frac{1}{2}}\phi\eta^{-\frac{1}{2}} \\ \mathrm{X} = \eta^{\frac{1}{2}}\phi\eta^{-\frac{1}{2}}, & \mathrm{X}^\times = \eta^{-\frac{1}{2}}\phi\eta^{\frac{1}{2}} \end{array}\right\}. \qquad (8)$$

From the list and from (8) the following statements, in which ξ_1, ξ_2 mean two linities of the type ξ, etc., are easily established :—

(1) $\xi_1\xi_2$ has no invariant significance, nor has $\xi_1{}^\times\xi_2{}^\times$.

(2) $\mathrm{X}_1\mathrm{X}_2$ is of type X and $\mathrm{X}_1{}^\times\mathrm{X}_2{}^\times$ is of type $\mathrm{X}^\times$.

(3) Although by (1), for integral values of c, ξ^c has in general no invariant significance; in particular when $c = -1$ it has significance. ξ^{-1} is of type $\xi^\times$, and $\xi^{\times-1}$ is of type ξ. X^{-1} is of type X, and $\mathrm{X}^{\times-1}$ is of type $\mathrm{X}^\times$.

(4) A rational function $f(\mathrm{X})$ is of type X; and $f(\mathrm{X}^\times)$ is of type $\mathrm{X}^\times$; the corresponding neutral form in each case being $f(\phi)$.

(5) ξ' is of type ξ, $\xi^{\times\prime}$ of type $\xi^\times$; so that a similar remark applies to the self-conjugate and skew parts of ξ and $\xi^\times$. ξ' and $\xi^{\times\prime}$ correspond to ϕ'.

(6) X' is of type $\mathrm{X}^\times$, $\mathrm{X}^{\times\prime}$ of type X, both corresponding to ϕ'. Again $\eta\mathrm{X}'\eta^{-1}$ is of type X and $\eta^{-1}\mathrm{X}^{\times\prime}\eta$ of type $\mathrm{X}^\times$, both corresponding to ϕ'.

(7) The identity satisfied by ξ or by $\xi^\times$ has no significance. The

identity satisfied by ϕ is the same as the identity satisfied by each of the following

$$\mathrm{X} = \xi\eta^{-1} = \eta\xi^{\times}, \qquad \mathrm{X}^{\times} = \eta^{-1}\xi = \xi^{\times}\eta,$$

and all their conjugates and η-conjugates.

(8) In the neutral form a finite rotation is effected by a vector linity (and its extensive) ϕ for which $\phi'\phi = 1$. This is equivalent to

$$\xi'\eta^{-1}\xi\eta^{-1} = \eta\mathrm{X}'\eta^{-1}\mathrm{X} = 1.$$

From the above it will be seen that if in a Galilean manifold we would deal with ϕ', $\phi' \pm \phi$, etc., then probably in the general manifold it is best to deal with ξ and $\xi^{\times}$; if in the Galilean we would use $\phi^{a}, f(\phi)$ etc., then in the general manifold the use of X, $\mathrm{X}^{\times}$ is indicated.

We need not write down the twenty-four alternative forms of the eight equations (6), but we may note how in terms of N, I the other types appear.

$$\left.\begin{array}{l} \mathrm{N}, \mathrm{I} \text{ are of type } \mathrm{X} \\ \mathrm{N}' = \eta^{-1}\mathrm{N}\eta, \quad \mathrm{I}' = \eta^{-1}\mathrm{I}\eta \text{ are of type } \mathrm{X}^{\times} \\ \mathrm{N}\eta, \mathrm{I}\eta \text{ are of type } \xi \\ \eta^{-1}\mathrm{N}, \eta^{-1}\mathrm{I} \text{ are of type } \xi^{\times} \end{array}\right\}. \qquad (9)$$

After the fundamental linity η, of type ξ, has presented itself the other three may be regarded as arising from

$$\xi^{-1}, \qquad \xi_1\xi_2^{-1}, \qquad \xi_1^{-1}\xi_2.$$

§ 14. *Riemann's and Weyl's Manifolds.*—Suppose ρ, $\rho+\alpha$, $\rho+\beta$ are for our manifold the position vectors of three infinitesimally near points, P, A, B. We assume that for such small regions the metrics are intrinsically the same as in a Galilean system, and in particular that there is a definite shift of PA along PB, and of PB along PA, each called a translation, by which if A in the first shift comes to C, B also in the second shift will come to C. With arbitrary co-ordinates we cannot of course affirm that C will have the position vector $\rho+\alpha+\beta$, but instead that it will have

$$\rho+\alpha+\beta+d_{\alpha}{}^{\mathrm{T}}\beta$$

where $d_{\alpha}{}^{\mathrm{T}}$ is to be regarded, not as a symbol of operation, but merely as the abbreviation of the words

$$d_{\alpha}{}^{\mathrm{T}} = \text{“ increment due to translation along } \alpha.\text{”} \qquad (1)$$

From these preliminaries it is clear that

$$d_{\alpha}{}^{\mathrm{T}}\beta = d_{\beta}{}^{\mathrm{T}}\alpha = -\mathrm{E}_{\alpha}\beta, \qquad (2)$$

where E_{α} is a vector linity which is a function of the position vector ρ, and is also a function of α, linear in the latter; also $\mathrm{E}_{\alpha}\beta$ is symmetrical in α and β.

The word translation implies something about constancy or otherwise in interval or in "the square measure" ds^2 of equation (2) of § 11. In Weyl's view we are to assume that in general *a translation is such* that it is impossible for the square measure to remain unaltered when by successive steps a closed path has been traced. In such a closed path beginning and ending at a given point, there will appear a change in the square measure depending on the point and the path.

The square measure then is assumed to change according to the equation

$$d_\alpha^{\mathrm{T}} \mathrm{V}_0 \beta\eta\beta = -\mathrm{V}_0\beta\eta\beta \,.\, \mathrm{V}_0\lambda\alpha, \tag{3}$$

where α and β are arbitrary and with the same meanings as before, and λ is a vector, a given function of ρ. Thus α, β must be contravariant and λ covariant. $\mathrm{E}_\alpha\beta$ is a vector neither covariant nor contravariant, which need not surprise us, because more than one position of the manifold has been used in its description.

It is clear that η is arbitrary as to a scalar factor and λ correspondingly arbitrary. More definitely, nothing is altered in the meanings given if (1) η is multiplied by a positive scalar h, any function of position, and (2) λ is altered to $\lambda + \nabla \log h$, as appears from equation (3).

Important qualification of the last sentence.—η here means the original *vector* linity denoted by η. Whenever we speak of altering gauge by multiplying η by h this restriction is to be understood. It is clear that ηw thereby changes, not to $h\eta w$, but to $h^c\eta w$.

"Into all expressions or formulæ, which exhibit analytically true metrical properties of the manifold, η and λ must enter in such a manner that invariance subsists (1) when there is arbitrary transformation of co-ordinates (ρ replaced by ρ'); and (2) when η, λ are replaced by $h\eta$, $\lambda + \nabla \log h$, whatever positive scalar function of ρ, h may be."*

Since (3) is true, and also what (3) becomes both when β is changed to γ and when changed to $\beta+\gamma$, we get the corresponding bilinear form

$$d_\alpha^{\mathrm{T}} \mathrm{V}_0 \beta\eta\gamma = -\mathrm{V}_0\beta\eta\gamma \,.\, \mathrm{V}_0\lambda\alpha, \tag{4}$$

where α, β, γ are all arbitrary. In the translation denoted by d_α^{T}, η suffers the increment $-\mathrm{V}_0\alpha\nabla \,.\, \eta$ and β and γ the increments $-\mathrm{E}_\alpha\beta$, $-\mathrm{E}_\alpha\gamma$. Hence

$$-\mathrm{V}_0\alpha\nabla_9 \,.\, \mathrm{V}_0\beta\eta_9\gamma - \mathrm{V}_0\mathrm{E}_\alpha\beta\eta\gamma - \mathrm{V}_0\beta\eta\mathrm{E}_\alpha\gamma = -\mathrm{V}_0\beta\eta\gamma \,.\, \mathrm{V}_0\lambda\alpha.$$

Put
$$\eta\mathrm{E}_\alpha = \Gamma_\alpha, \tag{5}$$

so that $\Gamma_\alpha\beta$, like $\mathrm{E}_\alpha\beta$, is symmetrical in α, β. Thus

$$-\mathrm{V}_0(\gamma\Gamma_\alpha\beta + \beta\Gamma_\alpha\gamma) = \mathrm{V}_0\alpha(\nabla_9 - \lambda) \,.\, \mathrm{V}_0\beta\eta_9\gamma.$$

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* Weyl, 'Raum, Zeit, Materie,' 3rd ed., p. 110. English translation, 1922, of 4th ed. (Brose), last sentence of p. 123.

Write down the two equations obtained by cyclically changing α, β, γ, change the signs of the second and third, and then add the three equations. We obtain

$$2V_0\alpha\Gamma_\beta\gamma = V_0\alpha\left[(\nabla_9-\lambda)V_0\beta\eta_9\gamma - V_0\beta(\nabla_9-\lambda)\,.\,\eta_9\gamma - V_0\gamma(\nabla_9-\lambda)\,.\,\eta_9\beta\right],$$

or

$$\Gamma_\beta\gamma = \tfrac{1}{2}\left[-V_0\beta(\nabla_9-\lambda)\,.\,\eta_9\gamma - V_0\gamma(\nabla_9-\lambda)\,.\,\eta_9\beta + (\nabla_9-\lambda)V_0\beta\eta_9\gamma\right]. \qquad (6)$$

For Riemann's manifold we have merely to put $\lambda = 0$. Since frequent reference must be made to this case of Riemann's manifold, we will put for the values of E_α and Γ_α when $\lambda = 0$, $\bar{E}_\alpha$ and $\bar{\Gamma}_\alpha$ respectively.

From (6)

$$(\Gamma_\beta+\Gamma_\beta')\gamma = -V_0\beta(\nabla_9-\lambda)\,.\,\eta_9\gamma, \qquad (7)$$

$$(\Gamma_\beta-\Gamma_\beta')\gamma = V_1\left[V_2(\nabla_9-\lambda)\eta_9\beta\,.\,\gamma\right]. \qquad (8)$$

Thus $\Gamma_\beta'\gamma$ is obtained from (6) by reversing the signs of the second and third terms on the right. It further follows that $\Gamma_\beta'\gamma$ regarded as a linity of β, instead of γ, is self-conjugate.

This, and its converse, can be shown to be true of any such function $\Upsilon_\alpha\beta$ which is symmetrical in α and β. Thus we may put for the ι_c component of $\Upsilon_\alpha\beta$, $\iota_c V_0\alpha\psi_c\beta$ where ψ_c is a self-conjugate vector linity, or

$$\left.\begin{aligned} \Upsilon_\alpha\beta &= \sum_{c=1}^{n} \iota_c V_0\beta\psi_c\alpha \\ \text{whence}\qquad \Upsilon_\alpha'\beta &= \sum_{c=1}^{n} \psi_c\alpha V_0\iota\beta \end{aligned}\right\}, \qquad (9)$$

showing that $\Upsilon_\alpha'\beta$ is self-conjugate with respect to α. Each ψ_c involves $\frac{1}{2}n(n+1)$ scalars, so that $\Upsilon_\alpha\beta$ involves $\frac{1}{2}n^2(n+1)$ scalars; when $n = 4$ this number is 40.

When $\Upsilon_\alpha\beta$ is not symmetrical (9) still holds, ψ_c now being not self-conjugate, and the number of scalars being n^3.

We shall from this point onwards take for granted, as is not difficult to prove, that *at a given point it is always possible so to choose our co-ordinates that all the* $\frac{1}{2}n^2(n+1)$ *differential coefficients* (∇_9, η_9) *are zero; and independently to choose the gauge* (h) *so that* λ *is zero; therefore also to make* E_α *zero for all values of* α. [Brose calls h the calibration ratio. Selection of a definite h he calls calibration.]

§ 15. *Absolute Differentiation.*—Let τ be a contravariant vector given over an assigned path or over a region of n or fewer dimensions. If the region is of fewer than n dimensions, we may still assign arbitrary values over a region of n dimensions containing the given region, so that we can continue to apply ∇ to it.

In passing from the point ρ to the point $\rho+\alpha$, where α is infinitesimal, the *ordinary* increment $-V_0\alpha\nabla.\tau$ will be denoted by $d\alpha^0\tau$, and the excess of this above $d_\alpha{}^{T}\tau$ will be called the absolute increment and denoted by $d_\alpha{}^{A}\tau$. Thus we have

$$\left.\begin{aligned} d_\alpha{}^{A}\tau &= d_\alpha{}^{0}\tau - d_\alpha{}^{T}\tau = d_\alpha{}^{0}\tau + E_\alpha\tau \\ &= -V_0\alpha\nabla\,.\,\tau + E_\alpha\tau \end{aligned}\right\}. \tag{1}$$

We are about to show, as might be expected, that this absolute increment, $d_\alpha{}^{A}\tau$, is contravariant. If

$$\chi = -V_0(\ )\nabla\,.\,\rho^{\backprime}, \tag{2}$$

and the change from ρ to $\rho^{\backprime}$ causes the multenion function q of position to change to $q^{\backprime}$, and the multenion linity ϕ to change to $\phi^{\backprime}$; then the necessary and sufficient conditions that q should be (1) contravariant, (2) covariant, are

$$(1)\ q^{\backprime} = \chi q, \qquad (2)\ q^{\backprime} = \chi'^{-1}q.$$

The necessary and sufficient conditions that ϕ should be (1) covariant, (2) contravariant, (3) comixt, (4) contramixt, are that $\phi^{\backprime}$ should be equal to

$$(1)\ \chi'^{-1}\phi\chi^{-1}, \qquad (2)\ \chi\phi\chi', \qquad (3)\ \chi'^{-1}\phi\chi', \qquad (4)\ \chi\phi\chi^{-1}.$$

All these may be put in infinitesimal forms which are generally easier to apply. Let

$$\rho^{\backprime} = \rho + \delta\rho = \rho + \epsilon, \qquad \chi^{\backprime} = 1 + \delta\chi = 1 + \chi_0,$$
$$q^{\backprime} = q + \delta q, \qquad \phi^{\backprime} = \phi + \delta\phi.$$

Then the above six conditions (necessary and sufficient) for the several types of invariance become

$$(1)\ \delta q = \chi_0 q, \qquad (2)\ \delta q = -\chi_0' q,$$

and in the case of ϕ, $\delta\phi$ is equal to

$$(1)\ -\chi_0'\phi - \phi\chi_0, \quad (2)\ \chi_0\phi + \phi\chi_0', \quad (3)\ -\chi_0'\phi + \phi\chi_0', \quad (4)\ \chi_0\phi - \phi\chi_0;$$

and in order to make the tests we have

$$\chi_0 = -V_0(\ )\nabla\,.\,\epsilon, \qquad \chi_0' = -\nabla_9 V_0(\ )\epsilon_9. \tag{3}$$

[χ is applied to general multenions by the rules applicable to extensives; χ_0 by those applicable to subextensives.]

First apply the test to $E_\alpha\tau$; is $\delta E_\alpha\tau = \chi_0 E_\alpha\tau$? Instead of working this directly it will be found easier first to find $\delta V_0\alpha\Gamma_\beta\gamma$ from the equation preceding (6) of § 14, in which observe that, since $V_0\alpha\nabla = V_0\alpha^{\backprime}\nabla^{\backprime}$, δ has no effect on $V_0\alpha\nabla$, $V_0\beta\nabla$, $V_0\gamma\nabla$. Further for α, β, γ,

$$\delta(\alpha, \beta, \gamma) = \chi_0(\alpha, \beta, \gamma),$$

and for η

$$\delta\eta = -\chi_0'\eta - \eta\chi_0.$$

It is thus easy to prove that

$$\delta V_0\alpha\Gamma_\beta\gamma = -V_0\beta\nabla_9\,.\,V_0\gamma\nabla_9\,.\,V_0\eta\alpha\epsilon_9,$$

and thence that

$$\delta \mathrm{E}_\beta \gamma = \chi_0 \mathrm{E}_\beta \gamma - \mathrm{V}_0 \beta \nabla \,.\, \mathrm{V}_0 \gamma \nabla \,.\, \epsilon, \tag{4}$$

so that $\mathrm{E}_a \tau$ is not contravariant. Next we have

$$\delta \,.\, d_a{}^0 \tau - \chi_0 d_a{}^0 \tau = -\delta (\mathrm{V}_0 \alpha \nabla \,.\, \tau) + \chi_0 (\mathrm{V}_0 \alpha \nabla \,.\, \tau)$$
$$= -\mathrm{V}_0 \alpha \nabla \,.\, \chi_0 \tau + \chi_0 (\mathrm{V}_0 \alpha \nabla \,.\, \tau)$$

because δ does not affect $\mathrm{V}_0 \alpha \nabla$, and $\delta \tau = \chi_0 \tau$. Thus

$$\delta \,.\, d_a{}^0 \tau = \chi_0 d_a{}^0 \tau + \mathrm{V}_0 \alpha \nabla_9 \,.\, \mathrm{V}_0 \tau \nabla_9 \,.\, \epsilon_9$$

whence finally

$$\delta d_a{}^{\mathrm{A}} \tau = \chi_0 d_a{}^{\mathrm{A}} \tau,$$

or $d_a{}^{\mathrm{A}} \tau$ is contravariant.

In one respect our taking of equation (1) to define $d_a{}^{\mathrm{A}} \tau$ has been misleading. If we follow on from it and equation (4) § 14 in a natural manner to define $d_a{}^{\mathrm{A}} \sigma^\times$ where $\sigma^\times$ is a covariant vector, the meaning arrived at will be on the whole inconvenient. For the moment we may take it as a sufficient reason for this statement that if equation (4) § 14 is to hold for any scalar invariant function of position, x, then $d_a{}^{\mathrm{T}} x$ being different from zero, if $d_a{}^{\mathrm{A}} x$ is to be equal to $d_a{}^0 x - d_a{}^{\mathrm{T}} x$, absolute and ordinary increments of invariants must differ. We reject the suggestion of equation (1), then, except for the increment of a *contravariant* vector. It is convenient to introduce a new increment, the invariantive increment, $d_a{}^{\mathrm{I}} q$. For τ we will take

$$d_a{}^{\mathrm{I}} \tau = d_a{}^{\mathrm{T}} \tau = -\mathrm{E}_a \tau, \tag{5}$$

but to fix the meaning for other cases we will take that

$$d_a{}^{\mathrm{I}} x = 0 \tag{6}$$

when x is an invariant scalar function of position. This gives

$$0 = d_a{}^{\mathrm{I}} \mathrm{V}_0 \sigma^\times \tau = \mathrm{V}_0 (d_a{}^{\mathrm{I}} \sigma^\times) \tau - \mathrm{V}_0 \sigma^\times \mathrm{E}_a \tau.$$

Since τ is arbitrary we get

$$d_a{}^{\mathrm{I}} \sigma^\times = \mathrm{E}_a{}' \sigma^\times. \tag{7}$$

Finally in all cases we put

$$d_a{}^{\mathrm{A}} = d_a{}^0 - d_a{}^{\mathrm{I}} = -\mathrm{V}_0 \alpha \nabla \,.\, - d_a{}^{\mathrm{I}}. \tag{8}$$

The strongest reason for this meaning of $d_a{}^{\mathrm{A}}$ remains to be given. It is the only increment of the kind which is *truly* invariantive for *truly* contravariant and covariant vectors; vectors which besides their usual relations to change of co-ordinates are also invariant for change of gauge. If τ is thus truly contravariant $\eta \tau$ cannot be truly covariant, and therefore we ought not to base our conception of the standard increment of $\sigma^\times$ on supposing it possible to put $\sigma^\times = \eta \tau$.

Nevertheless the following up of the meaning of $d_a{}^{\mathrm{T}} \sigma^\times$ from equation (4) of § 14 has its uses, and we shall make one important application of it when later we consider curvature. It is not difficult to show that

$$d_\alpha^{\mathrm{T}}\sigma^\times = \mathrm{E}_\alpha'\sigma^\times - \sigma^\times \mathrm{V}_0\alpha\lambda, \qquad d_\alpha^{\mathrm{I}}\eta = d_\alpha^0\eta = -\mathrm{V}_0\alpha\nabla\,.\,\eta \tag{9}$$

(so that if we had continued to take $d_\alpha^{\mathrm{A}} = d_\alpha^{\mathrm{o}} - d_\alpha^{\mathrm{T}}$ we should have found, as in the Riemann manifold, that $d_\alpha^{\mathrm{A}}\eta = 0$). With the meaning of (8) above $d_\alpha^{\mathrm{A}}\eta$ is not zero, as in a Riemann manifold, but instead

$$d_\alpha^{\mathrm{A}}\eta = -\mathrm{V}_0\alpha\lambda\,.\,\eta$$

when η is used in its primitive sense as a vector linity. When used as an extensive linity in ηw we get

$$d_\alpha^{\mathrm{A}}\eta\,.\,w = -c\mathrm{V}_0\alpha\lambda\,.\,\eta w. \tag{10}$$

We will now find the absolute increments of the two kinds of multenions and the four of multenion linities. Mixed linities interpreted as subextensive, have invariant meanings, but not the other two types. Thus if ϕ is contramixt

$$\phi\mathrm{V}_3\alpha\beta\gamma = \mathrm{V}_3\phi\alpha\beta\gamma + \mathrm{V}_3\alpha\phi\beta\gamma + \mathrm{V}_3\alpha\beta\phi\gamma$$

is of invariant form, because then $\phi\alpha$, β, γ are of one type, and is not of such form when ϕ is covariant, because then $\phi\alpha$, β, γ are not all of one type.

Let E_α and E_α' have the subextensive meanings as well as their primitive vector linity meanings. Thus

$$\mathrm{E}_\alpha w = -\mathrm{V}_c\mathrm{E}_\alpha\zeta\mathrm{V}_{c-1}\zeta w, \qquad \mathrm{E}_\alpha' w = -\mathrm{V}_c\mathrm{E}_\alpha'\zeta\mathrm{V}_{c-1}\zeta w. \tag{11}$$

We at once have

$$d_\alpha^{\mathrm{A}}p = d_\alpha^0 p - d_\alpha^{\mathrm{I}}p = -\mathrm{V}_0\alpha\nabla\,.\,p + \mathrm{E}_\alpha p, \tag{12}$$

$$d_\alpha^{\mathrm{A}}q^\times = d_\alpha^0 q^\times - d_\alpha^{\mathrm{I}}q^\times = -\mathrm{V}_0\alpha\nabla\,.\,q^\times - \mathrm{E}_\alpha' q^\times. \tag{13}$$

For the four linities $\xi_\times$, $\xi^\times$, X, $\mathrm{X}^\times$, treat of the first as an example of all four. The meaning of $d_\alpha^{\mathrm{A}}\xi$ is given by

$$d_\alpha^{\mathrm{A}}\xi\,.\,p = d_\alpha^{\mathrm{A}}(\xi p) - \xi d_\alpha^{\mathrm{A}}p.$$

Here ξp is a $q^\times$ so that

$$d_\alpha^{\mathrm{A}}(\xi p) = -\mathrm{V}_0\alpha\nabla\,.\,(\xi p) - \mathrm{E}_\alpha'\xi p.$$

Hence

$$d_\alpha^{\mathrm{A}}\xi\,.\,p = -\mathrm{V}_0\alpha\nabla_9\,.\,\xi_9 p - \mathrm{E}_\alpha'\xi p - \xi\mathrm{E}_\alpha p.$$

Thus

$$\left.\begin{aligned} d_\alpha^{\mathrm{A}}\xi &= -\mathrm{V}_0\alpha\nabla\,.\,\xi - \mathrm{E}_\alpha'\xi - \xi\mathrm{E}_\alpha \\ d_\alpha^{\mathrm{A}}\xi^\times &= -\mathrm{V}_0\alpha\nabla\,.\,\xi^\times + \mathrm{E}_\alpha\xi^\times + \xi^\times\mathrm{E}_\alpha' \\ d_\alpha^{\mathrm{A}}\mathrm{X} &= -\mathrm{V}_0\alpha\nabla\,.\,\mathrm{X} - \mathrm{E}_\alpha'\mathrm{X} + \mathrm{X}\mathrm{E}_\alpha' \\ d_\alpha^{\mathrm{A}}\mathrm{X}^\times &= -\mathrm{V}_0\alpha\nabla\,.\,\mathrm{X}^\times + \mathrm{E}_\alpha\mathrm{X}^\times - \mathrm{X}^\times\mathrm{E}_\alpha \end{aligned}\right\}. \tag{14}$$

It is instructive to re-write in the present notation the four kinds of $\delta\phi$ above

$$\left.\begin{aligned} \delta\xi &= -\chi_0{}'\xi-\xi\chi_0 \\ \delta\xi^\times &= \chi_0\xi^\times+\xi^\times\chi_0{}' \\ \delta \mathrm{X} &= -\chi_0{}'\mathrm{X}+\mathrm{X}\chi_0{}' \\ \delta \mathrm{X}^\times &= \chi_0\mathrm{X}^\times-\mathrm{X}^\times\chi_0 \end{aligned}\right\}. \tag{15}$$

Doubtless the complete parallelism between (14) and (15) has some simple explanation. Of the four results in each set, the second follows from the first by putting $\xi^\times = \xi^{-1}$ and the fourth from the third by putting $\mathrm{X}^\times = \mathrm{X}'$.

We obtain six very general formulæ showing how to convert such expressions as $(\nabla_9, q_9{}^\times)$ or (∇_9, p_9) to invariant form [provided $(\alpha^\times, q^\times)$ or $(\alpha^\times, p)$ are already bilinear and invariantive] by one common process. Introduce a second notation for $d_\alpha{}^\mathrm{A}$, thus

$$d_\alpha{}^\mathrm{A} q = q_\alpha, \qquad d_\alpha{}^\mathrm{A}\phi = \phi_\alpha, \text{ etc.}$$

Put ζ in place of α in (12), (13), and (14). Thus

$$(\zeta, p_\zeta) = (\nabla_9, p_9)+(\zeta, \mathrm{E}_\zeta p), \tag{12a}$$

$$(\zeta, q_\zeta{}^\times) = (\nabla_9, q_9{}^\times)-(\zeta, \mathrm{E}_\zeta{}' q^\times), \tag{13a}$$

$$\left.\begin{aligned} (\zeta, \xi_\zeta) &= (\nabla_9, \xi_9)+(\zeta, -\mathrm{E}_\zeta{}'\xi-\xi\mathrm{E}_\zeta) \\ (\zeta, \xi_\zeta{}^\times) &= (\nabla_9, \xi_9{}^\times)+(\zeta, \mathrm{E}_\zeta\xi^\times+\xi^\times\mathrm{E}_\zeta{}') \\ (\zeta, \mathrm{X}_\zeta) &= (\nabla_9, \mathrm{X}_9)+(\zeta, -\mathrm{E}_\zeta{}'\mathrm{X}+\mathrm{X}\mathrm{E}_\zeta{}') \\ (\zeta, \mathrm{X}_\zeta{}^\times) &= (\nabla_9, \mathrm{X}_9{}^\times)+(\zeta, \mathrm{E}_\zeta\mathrm{X}^\times-\mathrm{X}^\times\mathrm{E}_\zeta) \end{aligned}\right\}. \tag{14a}$$

(12), (13), (14) are particular cases of (12*a*), (13*a*), (14*a*).

As an example of the use of these equations let us suppose we have with Galilean co-ordinates established the equation $\mathrm{T}_9\nabla_9 = 0$ where T is the (complete) energy momentum linity; a linity which in the general manifold is best taken as comixt; a self-conjugate *vector* linity (self-conjugacy means for a *comixt* linity $\mathrm{T} = \eta\mathrm{T}'\eta^{-1}$, that is $\eta^{-1}\mathrm{T}$ or $\mathrm{T}\eta$ is self-conjugate in the ordinary sense). The (truly) invariant form of $\mathrm{T}_9\nabla_9$ is given as $\mathrm{T}_\zeta\zeta$ by the third of (14*a*). For generality we will at first not assume T to be self-conjugate. We have

$$\mathrm{T}_\zeta\zeta = \mathrm{T}_9\nabla_9-\mathrm{E}_\zeta{}'\mathrm{T}\zeta+\mathrm{T}\mathrm{E}_\zeta{}'\zeta.$$

To evaluate $\mathrm{E}_\zeta{}'\mathrm{T}\zeta$, write down $\mathrm{E}_\alpha{}'\beta^\times = \Gamma_\alpha{}'\eta^{-1}\beta^\times$ from (6) of § 14, change α, $\beta^\times$ to ζ, $\mathrm{T}\zeta$, and simplify. Thus

$$\mathrm{E}_\zeta{}'\mathrm{T}\zeta = \tfrac{1}{2}\,\eta_9\,(\eta^{-1}\mathrm{T}-\mathrm{T}'\eta^{-1})\,(\nabla_9-\lambda)-\tfrac{1}{2}\,(\nabla_9-\lambda)\,\mathrm{V}_0\zeta\eta_9\eta^{-1}\mathrm{T}\zeta. \tag{16}$$

When η^{-1} T is self-conjugate we get

$$\mathrm{E}_\zeta{}'\mathrm{T}\zeta = -\tfrac{1}{2}\,(\nabla_9-\lambda)\,\mathrm{V}_0\zeta\eta_9\eta^{-1}\mathrm{T}\zeta. \tag{17}$$

Further, since $|\phi|^{-1} . d|\phi| = -V_0\zeta\phi^{-1}d\phi\zeta$, putting $T = 1$ we get

$$E_\zeta'\zeta = k^{-1}\nabla k - \tfrac{1}{2}n\lambda. \tag{18}$$

In equation (16), the skew part of $\eta^{-1}T$ alone occurs in the first term, and the self-conjugate part alone in the second. $\eta^{-1}T$ is contravariant. Let us put $\eta^{-1}T = \xi^\times$ and use the second of (14*a*) for the further reduction

$$\xi_\zeta{}^\times\zeta = \xi_9{}^\times\nabla_9 + E_\zeta\xi^\times\zeta + \xi^\times E_\zeta'\zeta.$$

We may use (18) to modify $E_\zeta'\zeta$. $E_\zeta\xi^\times\zeta$ happens to be particularly simple as it involves only the self-conjugate part of $\xi^\times$; let $\xi^\times = \xi_0{}^\times + \xi_1{}^\times$ where $\xi_0{}^\times$ is the self-conjugate part and $\xi_1{}^\times$ the skew part. We find that

$$E_\zeta\xi^\times\zeta = \eta^{-1}(\eta_9\xi_0{}^\times + \tfrac{1}{2}V_0\zeta\eta_9\xi_0{}^\times\zeta)(\nabla_9 - \lambda) \tag{17a}$$

and $$\xi_\zeta{}^\times\zeta = k^{-1}(k\xi^\times)_9(\nabla_9 - \tfrac{1}{2}n\lambda) + \eta^{-1}(\eta_9\xi_0{}^\times + \tfrac{1}{2}V_0\zeta\eta_9\xi_0{}^\times\zeta)(\nabla_9 - \lambda). \tag{17b}$$

Linities of the form $\phi + \frac{1}{2}V_0\zeta\phi\zeta$ frequently present themselves in relativity, ϕ being mixed and generally self-conjugate in the mixed sense.

Note that $\xi_\zeta\zeta$ and $X_\zeta{}^\times\zeta$ have no invariant meaning because $\xi\beta^\times$ and $X^\times\beta^\times$ have none. We thus do not seek generalisations of $\xi_9\nabla_9$ and $X_9{}^\times\nabla_9$. ∇ is a symbolic *covariant* vector because $V_0 d\rho\nabla = V_0 d\rho'\nabla'$.

(17*a*) is found again below and numbered (22), in the course of illustrating a generally useful process.

Returning to the generalisation required of $T_9\nabla_9 = 0$, when T is comixt and η^{-1} T is self-conjugate, we get

$$(kT)_9(\nabla_9 - \tfrac{1}{2}n\lambda) + \tfrac{1}{2}(\nabla_9 - \lambda)V_0\zeta\eta_9\eta^{-1}(kT)\zeta = 0. \tag{19}$$

It is clear that kT, a vector linity *density* as it is called, rather than T itself, should be regarded as the fundamental function. (19) is our standard form on the hypothesis that T is *truly* invariantive. If it were of power a in the sense about to be explained, the first ∇_9 in (19) should be replaced by $\nabla_9 - a\lambda$.

If an invariantive symbol has such a meaning that on change of gauge ($h\eta$ for η, $\lambda + h^{-1}\nabla h$ for λ) the symbol is multiplied by h^a then it is said to be of power a. η is of power 1 when used in its primitive sense as a vector linity; in ηw, η is of power c; k is of power $\frac{1}{2}n$. It is because of these powers that we find when ∇ applies to η (power 1) as in E_a and Γ_a, ∇ appears only in the combination $\nabla - \lambda$; when ∇ applies to k (power $\frac{1}{2}n$) we always find it in the combination $\nabla - \frac{1}{2}n\lambda$. Thus in (18), although in a Riemann manifold we should find $E_\zeta'\zeta = k^{-1}\nabla k$, we must expect in the Weyl manifold to find instead

$$E_\zeta'\zeta = k^{-1}(\nabla - \tfrac{1}{2}n\lambda)k.$$

The reason for this is easily seen. Thus if q is of power a, (∇_9, q_9) is of no definite power at all, but as is quite easy to verify $(\nabla_9 - a\lambda,\ q_9)$ is of power a. To be *truly* invariantive a symbol must be of power zero.

An important case of this kind occurs in connection with the earliest invariant expressions we met, namely,

$$k^{-1}\mathrm{V}_{a-1}\nabla(ku), \qquad \mathrm{V}_{b+1}\nabla v^{\times}.$$

Our standard forms for these are based on the supposition that u and $v^{\times}$ are each of power zero so that $\mathrm{V}_{b+1}\nabla v^{\times}$ has not to be modified; but in the other case ku is of power $\frac{1}{2}n$ and our new invariant form should be $k^{-1}\mathrm{V}_{a-1}(\nabla - \frac{1}{2}n\lambda)(ku)$. Now (12$a$) and (13$a$) show us that our standard forms in these two cases should be

$$\mathrm{V}_{a-1}\nabla u + \mathrm{V}_{a-1}\zeta\mathrm{E}_{\zeta}u, \qquad \mathrm{V}_{b+1}\nabla v^{\times} - \mathrm{V}_{b+1}\zeta\mathrm{E}_{\zeta}'v^{\times}.$$

These must be the same as those we have just been considering so that we must have

$$\mathrm{V}_{a-1}\zeta\mathrm{E}_{\zeta}u = k^{-1}\mathrm{V}_{a-1}[(\nabla - \tfrac{1}{2}n\lambda)k \,.\, u], \tag{20}$$

$$\mathrm{V}_{b+1}\zeta\mathrm{E}_{\zeta}'v^{\times} = 0. \tag{21}$$

These identities can be established directly, but not very easily.

The specially simple form of $\Gamma_{\alpha} + \Gamma_{\alpha}'$ given in (7) of § 14 furnishes a second useful result from (16) for it gives

$$(\Gamma_{\zeta}' + \Gamma_{\zeta})\eta^{-1}\mathrm{T}\zeta = \eta_9\eta^{-1}\mathrm{T}(\nabla_9 - \lambda)$$

where $\Gamma_{\zeta}'\eta^{-1} = \mathrm{E}_{\zeta}'$. Thus from (16)

$$\Gamma_{\zeta}\eta^{-1}\mathrm{T}\zeta = -\tfrac{1}{2}\eta_9(\eta^{-1}\mathrm{T} + \mathrm{T}'\eta^{-1})(\nabla_9 - \lambda) + \tfrac{1}{2}(\nabla_9 - \lambda)\mathrm{V}_0\zeta\eta_9\eta^{-1}\mathrm{T}\zeta. \tag{22}$$

First we have the specially simple case $\mathrm{T} = 1$, namely

$$\Gamma_{\zeta}\eta^{-1}\zeta = \eta_9\eta^{-1}\nabla_9 - k^{-1}\nabla k + \tfrac{1}{2}(n-2)\lambda. \tag{23}$$

Another very simple particular case is when $\eta^{-1}\mathrm{T}$ is skew. The right of (22) is then zero, and $\eta^{-1}\mathrm{T}$ is of the form $\mathrm{V}_1\omega(\quad)$ where ω is a contravariant ${}_n\mathrm{V}_2$. Thus

$$\Gamma_{\zeta}\mathrm{V}_1\omega\zeta = 0.$$

Since $\Gamma_{\alpha} = \eta\mathrm{E}_{\alpha}$ we also have $\mathrm{E}_{\zeta}\mathrm{V}_1\omega\zeta = 0$. This last may be regarded as a special case of (21) and suggests two other forms of (20), (21). Operate on (20) by $\mathrm{V}_0 w^{\times}(\quad)$ taking $c = a-1$; and on (21) by $\mathrm{V}_0 w(\quad)$ taking $c = b+1$. We thus get

$$\mathrm{E}_{\zeta}'\mathrm{V}_{c+1}w^{\times}\zeta = k^{-1}\mathrm{V}_{c+1}w^{\times}(\nabla - \tfrac{1}{2}n\lambda)k, \tag{24}$$

$$\mathrm{E}_{\zeta}\mathrm{V}_{c-1}w\zeta = 0, \tag{25}$$

(18) is a particular case of (24).

§ 16. *Curvature.*—If (α, β) is any function of two vectors, linear in each then

$$(\alpha, \beta) - (\beta, \alpha) = (\zeta, \mathrm{V}_1[\mathrm{V}_2\alpha\beta \,.\, \zeta]) \tag{1}$$

as is seen by putting $(\alpha, \beta) = -(\zeta, \beta V_0 \alpha \zeta)$ and $(\beta, \alpha) = -(\zeta, \alpha V_0 \beta \zeta)$. Thus when

$$(\alpha, \beta) = -(\beta, \alpha) \quad \text{then} \quad (\alpha, \beta) = \tfrac{1}{2}(\zeta, V_1 [V_2 \alpha\beta \,.\, \zeta]). \tag{2}$$

The application of these two equations should be constantly borne in mind in what follows, even when attention is not called to it.

Similarly for the various invariant forms of infinitesimal rotations of a vector,

$$V_1 \omega \alpha^{\times}, \qquad V_1 \omega^{\times} \alpha, \qquad V_1 \omega \eta \alpha, \qquad V_1 \omega^{\times} \eta^{-1} \alpha^{\times}.$$

Let us calculate the increase in γ, a contravariant vector, due to translation, starting from the point ρ, along the four vector sides of an infinitesimal parallelogram in the order α, β, $-\alpha$, $-\beta$, α and β being contravariant. Afterwards let us calculate the similar increments of $\eta\gamma$. We shall put $V_2 \alpha\beta = \omega$, and the increments due to translation round the parallelogram will be denoted by $d_\omega^{T}\gamma$, $d_\omega^{T}\eta\gamma$. Since the difference of two contravariant vectors at a point ρ is itself contravariant it follows that $d_\omega^{T}\gamma$ is contravariant, and similarly $d_\omega^{T}\eta\gamma$ is covariant; and it is obvious that they must both be linear in both ω and γ. We shall, therefore, put

$$d_\omega^{T}\eta\gamma = \eta d_\omega^{T}\gamma = \eta C_\omega \gamma = \Omega_\omega \gamma, \tag{3}$$

C_ω being a contramixt vector linity and Ω_ω the corresponding covariant linity, which may be called the curvature linities. Along the side $+\alpha$ the increment in γ is $-E_\alpha \gamma$. Along the side $-\alpha$, γ has changed to $\gamma - E_\beta \gamma$ and $d_{-\alpha}^{T} = -d_\alpha^{T}$ has received the increment $-V_0 \beta \nabla . (\quad)$. Thus, these two sides contribute increments

$$-E_\alpha \gamma + (E_\alpha - V_0 \beta \nabla \,.\, E_\alpha)(\gamma - E_\beta \gamma),$$
$$= -V_0 \beta \nabla \,.\, E_\alpha \gamma - E_\alpha E_\beta \gamma.$$

The contributions of the other two sides are obtained by interchange of α and β and reversal of the sign. Hence, noting that E_α is linear in α,

$$C_\omega \gamma = d_\omega^{T}\gamma = -E_{9\,.\,V_1 \omega \nabla_9} \gamma + (-E_\alpha E_\beta + E_\beta E_\alpha)\gamma. \tag{4}$$

[Since $d_\alpha^{A} = d_\alpha^{0} - d_\alpha^{T}$, and since d_α^{0} taken round the parallelogram produces zero we get that $C_\omega \tau = \tau_{\alpha\beta} - \tau_{\beta\alpha}$ which shows that absolute differentiations in different directions are not commutative.]

In treating of $\eta\gamma$ in place of γ, instead of $-E_\alpha$ we have to write $E_\alpha' - V_0 \lambda \alpha$ for d_α^{T}. Noting that if x, y are two scalars $(-V_0 \lambda \alpha, -V_0 \lambda \beta)$

$$-(E_\alpha' + x)(E_\beta' + y) + (E_\beta' + y)(E_\alpha' + x) = -E_\alpha' E_\beta' + E_\beta' E_\alpha'$$

we see at once how (4) has to be modified. Thus

$$\Omega_\omega \gamma = d_\omega^{T}\eta\gamma = E_{9\,.\,V_1 \omega \nabla_9}' \eta\gamma + (-E_\alpha' E_\beta' + E_\beta' E_\alpha')\eta\gamma - \eta\gamma \,.\, V_0 \omega\, \omega^{\times}, \tag{5}$$

where

$$\omega^{\times} = V_2 \nabla \lambda \tag{6}$$

[ω and $\omega^{\times}$ are of very different natures. ω is a contravariant *dummy*, $\omega^{\times}$ is a given truly covariant function of position; in relativity $\omega^{\times}$ is magnetic induction cum electrostatic intensity, λ being vector potential cum electrostatic potential.]

From (4) on putting $\omega = \mathrm{V}_2\alpha\beta$, and noting that $\mathrm{E}_\alpha\gamma$ is symmetrical in α and γ, the following "cyclic rule" is seen to be true for each of the two parts of $\mathrm{C}_\omega\gamma$ separately

$$\mathrm{C}_{\mathrm{V}_2\alpha\beta}\gamma + \mathrm{C}_{\mathrm{V}_2\beta\gamma}\alpha + \mathrm{C}_{\mathrm{V}_2\gamma\alpha}\beta = 0. \tag{7}$$

Remembering that $\Omega_\omega = \eta\mathrm{C}_\omega$, inspection of (4) and (5) shows that

$$\left.\begin{aligned} \Omega_\omega + \Omega_\omega' &= -\mathrm{V}_0\omega\omega^{\times} \,.\, \eta \\ \text{or} \qquad \mathrm{C}_\omega + \eta^{-1}\mathrm{C}_\omega'\eta &= -\mathrm{V}_0\omega\omega^{\times} \end{aligned}\right\}. \tag{8}$$

The equation in $\Omega_\omega + \Omega_\omega'$ is even simpler than that in $\Gamma_\alpha + \Gamma_\alpha'$; and for a Riemann manifold $\Omega_\omega + \Omega_\omega' = 0$.

The first of (8) shows that $\Omega_\omega = -\frac{1}{2}\mathrm{V}_0\omega\omega^{\times}.\eta + \mathrm{V}_1\omega'(\quad)$ where ω' is a covariant linear function of ω. Hence

$$\Omega_\omega\gamma = -\tfrac{1}{2}\mathrm{V}_0\omega\omega^{\times} \,.\, \eta\gamma + \mathrm{V}_1\varpi\omega\gamma, \tag{9}$$

where ϖ is a covariant ${}_n\mathrm{V}_2$ linity. The first term on the right of (9) gives the ratio of stretch $-\frac{1}{2}\mathrm{V}_0\omega\omega^{\times}$; the second term the rotation $\varpi\omega$; suffered by γ by $d_\omega{}^{\mathrm{T}}$.

To find the properties of the linity ϖ first transform (5) to the following,

$$\Omega_\omega\gamma = \Gamma_{9\,.\,\mathrm{V}_1\omega(\nabla_9-\lambda)}'\gamma + (\Gamma_\alpha'\eta^{-1}\Gamma_\beta - \Gamma_\beta'\eta^{-1}\Gamma_\alpha)\gamma - \eta\gamma \,.\, \mathrm{V}_0\omega\omega^{\times}. \tag{10}$$

I will indicate in outline the mode of doing this. The first term on the right of (5) is

$$(\Gamma_\alpha'\eta^{-1})_9\,\eta\gamma\mathrm{V}_0\beta\nabla_9 - (\Gamma_\beta'\eta^{-1})_9\,\eta\gamma\mathrm{V}_0\alpha\nabla_9$$

and the second term is

$$(-\Gamma_\alpha'\eta^{-1}\Gamma_\beta' + \Gamma_\beta'\eta^{-1}\Gamma_\alpha')\gamma = (\Gamma_\alpha'\eta^{-1}\Gamma_\beta - \Gamma_\beta'\eta^{-1}\Gamma_\alpha)\gamma + \text{two terms}$$

of which the first is $-\Gamma_\alpha'\eta^{-1}(\Gamma_\beta + \Gamma_\beta')\gamma$ and the second is $+\Gamma_\beta'\eta^{-1}(\Gamma_\alpha + \Gamma_\alpha')\gamma$. It will now be found that the parts depending on ∇ in $(\Gamma_\beta + \Gamma_\beta')$ and $(\Gamma_\alpha + \Gamma_\alpha')$ cancel with the following parts of the first term.

$$\Gamma_\alpha'(\eta^{-1})_9\eta\gamma \,.\, \mathrm{V}_0\beta\nabla_9 - \Gamma_\beta'(\eta^{-1})_9\eta\gamma \,.\, \mathrm{V}_0\alpha\nabla_9.$$

Collecting the parts not cancelled we get (10).

In the first term and the third on the right of (10) there are terms of the form (∇_9, λ_9). These give

$$\begin{aligned} &-\tfrac{1}{2}\eta\gamma \,.\, \mathrm{V}_0\omega\omega^{\times} - \tfrac{1}{2}\mathrm{V}_1 \,.\, \mathrm{V}_2(\eta\mathrm{V}_1\omega\nabla_9 \,.\, \lambda_9)\gamma \\ &= -\tfrac{1}{2}\eta\gamma \,.\, \mathrm{V}_0\omega\omega^{\times} - \tfrac{1}{4}\mathrm{V}_1 \,.\, \mathrm{V}_2(\eta\mathrm{V}_1\omega\nabla_9 \,.\, \lambda_9 + \eta\mathrm{V}_1\omega\lambda_9 \,.\, \nabla_9)\gamma \\ &\qquad - \tfrac{1}{4}\mathrm{V}_1 \,.\, \mathrm{V}_2(\eta\mathrm{V}_1\omega\nabla_9 \,.\, \lambda_9 - \eta\mathrm{V}_1\omega\lambda_9 \,.\, \nabla_9)\gamma. \end{aligned}$$

The last of these three parts furnishes the skew part ϖ_1, of ϖ, given by

$$\varpi_1\omega = \tfrac{1}{4}\mathrm{V}_2(\mathrm{V}_1\zeta\omega^{\times} \,.\, \eta\mathrm{V}_1\zeta\omega). \tag{11}$$

The neutral form of this is the suggestive expression $\frac{1}{4}V_2\omega^\times\omega$. The remainder of ϖ, denoted by ϖ_0, the self-conjugate part, is

$$\left.\begin{array}{l}\varpi_0\omega = -\frac{1}{2}V_2[(\nabla_9-\lambda)\eta_9V_1\omega(\nabla_9-\lambda)]+\frac{1}{2}V_2\zeta_0\Gamma_\zeta'\eta^{-1}\Gamma_{\zeta_0}V_1\omega\zeta \\ \qquad -\frac{1}{4}V_2(\eta V_1\omega\nabla_9 \,.\, \lambda_9+\eta V_1\omega\lambda_9 \,.\, \nabla_9)\end{array}\right\}. \tag{12}$$

So far we have divided $\Omega_\omega\gamma$ into three parts, such that the corresponding parts of $C_\omega\gamma$ are *truly* invariant, (1) the part self-conjugate in γ, namely, $-\frac{1}{2}\eta\gamma V_0\omega\omega^\times$; (2) the part $V_1\varpi_1\omega\gamma$ in which ϖ_1 is skew and (apart from the fundamental linity η) only involves $\omega^\times$ as with the first part; (3) the part $V_1\varpi_0\omega\gamma$ where ϖ_0 is self-conjugate. Part (3) obeys the cyclic rule; (1) and (2) together, but not separately, obey the same. The cyclic rule for ϖ_0 may be put, as will be shown later, in the form

$$V_4 \,.\, V_2\zeta_1\zeta_2\varpi_0V_2\zeta_1\zeta_2 = 0,$$

so that ϖ_0 instead of having the full number

$$\tfrac{1}{2} \,.\, \tfrac{1}{2}n(n-1) \,.\, [\tfrac{1}{2}n(n-1)+1]$$

of scalars appropriate to a self-conjugate ${}_nV_2$ linity, has a smaller number because of the $n(n-1)(n-2)(n-3)/4\,!$ scalar equations of the cyclic condition just written. The number of scalars of ϖ_0 is thus $\frac{1}{12}n^2(n^2-1)$. For the parts (1) and (2) the $\frac{1}{2}n(n-1)$ scalars of $\omega^\times$ are required. The total number of scalars involved in C_ω is thus

$$\tfrac{1}{12}n(n-1)(n^2+n+6).$$

When $n = 4$ the number is 26.

Putting $V_1\zeta\omega = \zeta_\omega$, $V_1\zeta\omega^\times = \zeta_{\omega\times}$ the two parts (1) and (2) together may be written as

$$\left.\begin{array}{l}-\frac{1}{4}\eta\gamma V_0\zeta_\omega\zeta_{\omega\times}+\frac{1}{4}V_1V_2(\zeta_{\omega\times}\eta\zeta_\omega)\gamma \\ = \frac{1}{4}(-\eta\gamma \,.\, V_0\zeta_\omega\zeta_{\omega\times}-\eta\zeta_\omega \,.\, V_0\zeta_{\omega\times}\gamma+\zeta_{\omega\times}V_0\zeta_\omega\gamma\end{array}\right\}. \tag{13}$$

It is now desirable to separate ϖ_0 into three parts ϖ_a, ϖ_b, ϖ_c, which are covariant as to change of co-ordinates though not as to change of gauge, namely, the parts of degrees zero, one, two in λ.

ϖ_a, of course, gives precisely the Riemann terms, obtained by putting $\lambda = 0$ in any expression for ϖ. Thus from (12)

$$\varpi_a\omega = -\tfrac{1}{2}V_2 \,.\, \nabla_9\eta_9V_1\omega\nabla_9+\tfrac{1}{2}V_2\zeta_0\bar{\Gamma}_\zeta'\eta^{-1}\bar{\Gamma}_{\zeta_0}V_1\omega\zeta. \tag{14}$$

Next write down the terms involving differentiations of λ and make them invariantive by (13*a*) of § 15, using, however, $\bar{E}_\zeta'$ in place of E_ζ' as we desire invariance only for change of co-ordinates, and we desire *not* to introduce quadratic terms in λ. Thus we obtain for the probable value of ϖ_b

$$\left.\begin{array}{l}\varpi_b\omega = -\frac{1}{4}V_2(\eta V_1\omega\nabla_9 \,.\, \lambda_9+\eta V_1\omega\lambda_9 \,.\, \nabla_9) \\ \qquad +\frac{1}{4}V_2(\eta V_1\omega\zeta \,.\, \bar{E}_\zeta'\lambda+\eta V_1\omega\bar{E}_\zeta'\lambda \,.\, \zeta)\end{array}\right\}. \tag{15}$$

Before verifying that this contains all the terms linear in λ, not differentiated, find the quadratic terms, $\varpi_c\omega$. Put Λ_a for the terms of Γ_a which contain λ, that is, $\Lambda_a = \Gamma_a - \overline{\Gamma}_a$. Thus

$$\varpi_c\omega = -\tfrac{1}{2}V_2\lambda\eta V_1\omega\lambda + \tfrac{1}{2}V_2\zeta_0\Lambda_\zeta'\eta^{-1}\Lambda_{\zeta_0}V_1\omega\zeta. \tag{16}$$

To simplify this put $\omega = V_2\alpha\beta$ and work in the neutral form, finally returning to invariant form. The result comes out that $4\varpi_c\omega$ is the normal expression of ω with reference to λ (in neutral form $\lambda V_3\lambda\omega$). Hence

$$\varpi_c\omega = \tfrac{1}{4}\eta V_2\lambda V_3\eta^{-1}\lambda\omega = \tfrac{1}{4}V_2\eta^{-1}\lambda V_3\lambda\eta\omega, \tag{17}$$

ϖ_a, ϖ_b, ϖ_c are all separately self-conjugate, and all separately obey the cyclic rule.

We will now show that (15) must contain all the terms of (12) which are linear in λ. Inspection of (12) shows that these terms form a function $(\nabla_9, \eta_9, \lambda, \omega)$ which is linear in all four constituents. If there is any term of this nature which has not been included in our three parts ϖ_a, ϖ_b, ϖ_c, such term must be covariant as to change of co-ordinates. Now $(\nabla_9, \eta_9, \lambda, \omega)$ can only be made covariant in this sense by the addition [first of (14*a*) of §15] to it of

$$(\zeta, -\overline{E}_\zeta'\eta - \eta\overline{E}_\zeta, \lambda, \omega) = -(\zeta, \overline{\Gamma}_\zeta' + \overline{\Gamma}_\zeta, \lambda, \omega) = -(\nabla_9, \eta_9, \lambda, \omega).$$

This addition reduces the supposed absent term to zero. Hence there is no such term.

We now proceed to the two successive "contractions" of Ω_ω or C_ω. The first contraction yields a vector linity denoted by Ω or C in its covariant or contramixt form. The second contraction yields a scalar, an invariant of course, F.

$\Omega_{V_2\zeta\beta}\eta^{-1}\zeta$ is a covariant vector because the subscript ζ occupies the place of a contravariant vector and the second ζ occupies the place of a covariant vector. Put then

$$\Omega\beta = \Omega_{V_2\zeta\beta}\eta^{-1}\zeta, \qquad C\beta = \eta^{-1}\Omega\beta = C_{V_2\zeta\beta}\eta^{-1}\zeta, \tag{18}$$

$$F = -V_0\zeta\eta^{-1}\Omega\zeta = -V_0\zeta C\zeta. \tag{19}$$

Note, that of the four symbols Ω_ω, C_ω, Ω, C, *the only two which are truly invariantive are* C_ω and Ω; Ω_ω is of power 1 and C of power -1; F also is of power -1.

Let us now obtain the terms contributed to $\Omega\beta$ and to F by our five parts of Ω_ω, that is, by the first term in (9) and the four parts of ϖ, namely, ϖ_1, ϖ_a, ϖ_b, ϖ_c. For the latter four other than ϖ_a work details in the neutral form, and at the last step put the result into invariant form by the rules of § 12. Where first differential coefficients of λ occur, attend in the detailed working only to the terms in which the differential coefficients occur, and in the last step use equation (13a), of § 15, modified by reading $\bar{E}_\zeta'$ instead of E_ζ' to obtain the invariant form.

Let us denote the parts contributed by the first term of equation (9), to Ω and F by Ω_2 and F_2, and those contributed by ϖ_1, ϖ_a, ϖ_b, ϖ_c, by Ω_1, Ω_a, Ω_b, Ω_c and by F_1, F_a, F_b, F_c, respectively. Thus we find

$$\left.\begin{aligned} \Omega_2\beta = -\tfrac{1}{2}V_1\omega^\times\beta, \qquad \Omega_1\beta &= \tfrac{1}{4}(n-2)\,V_1\omega^\times\beta \\ (\Omega_2+\Omega_1)\,\beta &= \tfrac{1}{4}(n-4)\,V_1\omega^\times\beta \end{aligned}\right\}. \tag{20}$$

These are the only parts of $\Omega\beta$ that are not self-conjugate, and they are purely skew. It is a remarkable fact that Ω is self-conjugate for the case $n = 4$, and for no other value of n. Of course, this accidental circumstance, if we are to call it such, is fortunate for the theory of relativity. From (20), it at once follows that

$$F_2 = F_1 = 0. \tag{21}$$

It is easy to show that when a term ϖ_x of ϖ is self-conjugate the corresponding part of Ω_x of Ω is self-conjugate. ϖ_a, ϖ_b, ϖ_c, are self-conjugate. We find

$$\left.\begin{aligned} 4\Omega_b\beta &= -2\eta\beta\,.\,k^{-1}V_0\nabla(k\eta^{-1}\lambda) - (n-2)(\nabla_9V_0\lambda_9\beta+\lambda_9V_0\nabla_9\beta+2\bar{E}_\beta'\lambda) \\ F_b &= -(n-1)\,k^{-1}V_0\nabla\,(k\eta^{-1}\lambda) \end{aligned}\right\}. \tag{22}$$

In Ω_c the normal expression of β with respect to λ occurs thus:

$$\left.\begin{aligned} \Omega_c\beta &= \tfrac{1}{4}(n-2)\,V_1\eta^{-1}\lambda V_2\lambda\eta\beta \\ F_c &= \tfrac{1}{4}(n-1)(n-2)\,V_0\lambda\eta^{-1}\lambda \end{aligned}\right\}. \tag{23}$$

In a Riemann manifold ϖ_a completely specifies Ω_ω, and therefore Ω, $\Omega_{\omega a}$, and Ω_a are given by any expressions for Ω_ω and Ω by putting $\lambda = 0$. The most useful form of Ω_a is derived from (5). To get $\Omega\beta$ from (5) we have to put $\alpha = \zeta$, $\eta\gamma = \zeta$. Thus we obtain

$$\Omega\beta = E_{\beta 9}'\nabla_9 - E_\zeta'E_\beta'\zeta + E_\beta'E_\zeta'\zeta + V_0\beta\nabla\,.\,E_\zeta'\zeta + V_1\beta\omega^\times. \tag{24}$$

Here note that the first three terms are just those, when we put $E_\beta' = T$, that were under discussion in equations (16), (17), (18) of § 15, and (24) may be modified accordingly. Our immediate purpose, however, is to put λ, and therefore $\omega^\times$ zero, so as to obtain Ω_a. Noting that $\bar{E}_\zeta'\zeta = k^{-1}\nabla k$ by (18) of § 15, we have

$$\Omega_a\beta = \bar{E}_{\beta 9}'\nabla_9 - \bar{E}_\zeta'\bar{E}_\beta'\zeta + (\bar{E}_\beta' + V_0\beta\nabla\,.\,)\,\nabla\log k. \tag{25}$$

This is the form (26·3) given in Eddington's Report on Relativity, that is to say, Eddington's G_{bc} is our $V_0\iota_b\Omega_a\iota_c$ [see equation (7), § 17 below]. (25) gives

$$F_a = -V_0\eta^{-1}\zeta(\bar{E}_{\zeta 9}'\nabla_9 - \bar{E}_{\zeta_0}'\bar{E}_\zeta'\zeta_0 + [\bar{E}_\zeta' + V_0\zeta\nabla\,.\,])\,\nabla\log k. \tag{26}$$

Returning to equation (24) let us find a useful expression for $C_a\beta$. It is convenient on occasion to use the following notations

$$\eta^{-1}\Gamma_\beta\eta^{-1} = \Gamma_\beta^\times, \qquad \eta^{-1}\bar{\Gamma}_\beta\,\eta^{-1} = \bar{\Gamma}_\beta^\times,$$

but this is a little cumbrous for our present purpose. Let us put

$$\overline{\Gamma}_\beta{}^{\times\prime} = \eta^{-1}\overline{\mathrm{E}}_\beta{}' = \xi_\beta{}^\times = \mu_\beta{}^\times + \nu_\beta{}^\times, \tag{24a}$$

where $\mu_\beta{}^\times$ and $\nu_\beta{}^\times$ [see (24*c*) below] are put for the self-conjugate and skew parts of $\xi_\beta{}^\times$, and avoid the double suffix notation $\xi_{\beta 0}{}^\times$, $\xi_{\beta 1}{}^\times$. Putting $\lambda = 0$ in (24) we have

$$\mathrm{C}_a\beta = \eta^{-1}(\eta\xi_\beta{}^\times)_9\nabla_9 - \eta^{-1}\overline{\Gamma}_\zeta{}'\xi_\beta{}^\times\zeta + \xi_\beta{}^\times\nabla\log k + \eta^{-1}(\mathrm{V}_0\beta\nabla\,.\,\nabla\log k)$$

$$[\text{Put } -\overline{\Gamma}_\zeta{}' = \overline{\Gamma}_\zeta{}' - (\overline{\Gamma}_\zeta + \overline{\Gamma}_\zeta{}') = \overline{\Gamma}_\zeta + \mathrm{V}_0\zeta\nabla\,.\,\eta.]$$

$$= \xi_{\beta 9}{}^\times\nabla_9 + \overline{\mathrm{E}}_\zeta\xi_\beta{}^\times\zeta + k^{-1}\xi_\beta{}^\times\nabla k + \eta^{-1}(\mathrm{V}_0\beta\nabla\,.\,\nabla\log k),$$

whence, using (17*a*) of § 15 for $\overline{\mathrm{E}}_\zeta\xi_\beta{}^\times\zeta$, which involves $\mu_\beta{}^\times$ but not $\nu_\beta{}^\times$,

$$\mathrm{C}_a\beta = k^{-1}(k\xi_\beta{}^\times)_9\nabla_9 + \eta^{-1}(\eta_9\mu_\beta{}^\times + \tfrac{1}{2}\mathrm{V}_0\zeta\eta_9\mu_\beta{}^\times\zeta)\nabla_9 + \eta^{-1}(\mathrm{V}_0\beta\nabla\,.\,\nabla\log k). \tag{24b}$$

$\mu_\beta{}^\times$ and $\nu_\beta{}^\times$ being $\frac{1}{2}\eta^{-1}(\pm\Gamma_\beta + \Gamma_\beta{}')\eta^{-1}$ are written down from (7) and (8) of § 14. Thus

$$\left.\begin{aligned}\mu_\beta{}^\times\gamma^\times &= \tfrac{1}{2}\mathrm{V}_0\beta\nabla_8\,.\,(\eta^{-1})_8\gamma^\times\\ \nu_\beta{}^\times\gamma^\times &= -\tfrac{1}{2}\mathrm{V}_1[\eta^{-1}\mathrm{V}_2\nabla_8\eta_8\beta\,.\,\gamma^\times]\end{aligned}\right\}. \tag{24c}$$

From (24*b*) we may write down $k\mathrm{F}_a$ in the form $-k\mathrm{V}_0\zeta\mathrm{C}_a\zeta$ and we shall reach an expression useful for reducing, later, $\delta(k\mathrm{F}_a)$ resulting from changing η to $\eta + \delta\eta$.

In what immediately follows let us work with the neutral forms, only occasionally stopping to point out the slightly more complicated invariant forms. In the case of a vector linity, ϕ, we may say that there is a definite part, namely, $-n^{-1}\mathrm{V}_0\zeta\phi\zeta$, which gives rise to the contraction $-\mathrm{V}_0\zeta\phi\zeta$; the second part, $\phi + n^{-1}\mathrm{V}_0\zeta\phi\zeta$, has zero contraction and involves only $n^2 - 1$ scalars as against the full number, n^2, for a vector linity. Similarly, it is possible to separate out from any vector expression (ω, γ) linear in each constituent a definite part $(n-1)^{-1}(\mathrm{V}_2\zeta\mathrm{V}_1\omega\gamma, \zeta)$ absorbing (generally) n^2 scalars from the full number, $\frac{1}{2}n^3(n-1)$ required to specify (ω, γ). The remainder of (ω, γ), namely

$$(\omega, \gamma) - (n-1)^{-1}(\mathrm{V}_2\zeta\mathrm{V}_1\omega\gamma, \zeta)$$

has zero contraction. The part thus furnishing the contraction C of C_ω is

$$(n-1)^{-1}\mathrm{C}_{\mathrm{V}_2\zeta\mathrm{V}_1\omega\gamma}\zeta = (n-1)^{-1}\mathrm{C}\mathrm{V}_1\omega\gamma$$

and it obviously satisfies the cyclic rule (and would do so whether $\mathrm{C}_\omega\gamma$ itself did so or not).

The following form of the cyclic rule for $\mathrm{C}_\omega\gamma$ may be noticed:—

$$\mathrm{C}_{\mathrm{V}_2\zeta\mathrm{V}_3\omega\gamma}\zeta = 0. \tag{27}$$

This is true because by (1) of § 7

$$\mathrm{V}_2\zeta\mathrm{V}_3\omega\gamma = \mathrm{V}_2\beta\gamma\mathrm{V}_0\alpha\zeta + \mathrm{V}_2\gamma\alpha\mathrm{V}_0\beta\zeta + \mathrm{V}_2\alpha\beta\mathrm{V}_0\gamma\zeta$$

(27) is closely connected with a generalisation of (1) and (2) above. The cyclic rule is an identity which is combinatorial in α, β, γ; that is to say, the sign of the expression equated to zero alters by interchange of any two of the three α, β, γ. If $(\alpha_1 \alpha_2, \ldots, \alpha_a)$ is any expression linear in each vector and combinatorial in the vectors, it can always be expressed as a linear function of $\alpha_a{}^{(a)}$. It is actually equal to

$$(a\,!)^{-1}(\zeta_1, \zeta_2, \ldots, \zeta_{a-1}, \mathrm{V}_1 \alpha_a{}^{(a)} \zeta_{a-1}{}^{(a-1)}), \tag{28}$$

(28) may be proved by mathematical induction by proving the general connecting link in the following successive equalities, a being, for example, taken as equal to 4,

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) &= (2\,!)^{-1}(\zeta, \mathrm{V}_1 \,.\, \mathrm{V}_2\alpha\beta \,.\, \zeta, \gamma, \delta)\\ &= (3\,!)^{-1}(\zeta_1, \zeta_2, \mathrm{V}_1 \,.\, \mathrm{V}_3\alpha\beta\gamma \mathrm{V}_2\zeta_1\zeta_2, \delta)\\ &= (4\,!)^{-1}(\zeta_1, \zeta_2, \zeta_3, \mathrm{V}_1 \,.\, \mathrm{V}_4\alpha\beta\gamma\delta \mathrm{V}_3\zeta_1\zeta_2\zeta_3).\end{aligned}$$

This is readily done when it is noticed that

$$\mathrm{V}_1\zeta_3{}''' \mathrm{V}_4\alpha\beta\gamma\delta = \Sigma \pm \alpha \mathrm{V}_0\zeta_3{}''' \beta\gamma\delta$$

and the like $(\zeta_3{}''' = \mathrm{V}_3\zeta_1\zeta_2\zeta_3)$.

Since $\qquad \mathrm{V}_2\zeta \mathrm{V}_1\omega\gamma - \mathrm{V}_2(\mathrm{V}_3\zeta\omega)\gamma = -\omega \mathrm{V}_0\zeta\gamma,$

we have $\qquad \mathrm{C}_\omega\gamma = \mathrm{C}\mathrm{V}_1\omega\gamma - \mathrm{C}_{\mathrm{V}_2(\mathrm{V}_3\zeta\omega)\gamma}\zeta \qquad (29)$

whether or not $\mathrm{C}_\omega\gamma$ satisfies the cyclic rule. If the rule is satisfied we have further

$$\mathrm{C}_\omega\gamma = \mathrm{C}_{\mathrm{V}_2(\mathrm{V}_1\zeta\omega)\gamma}\zeta. \tag{30}$$

This may be obtained from (29) or directly from the cyclic rule in its original form thus:

$$\mathrm{C}_\omega\gamma = \mathrm{C}_{\mathrm{V}_2\alpha\beta}\gamma = \mathrm{C}_{\mathrm{V}_2\alpha\gamma}\beta - \mathrm{C}_{\mathrm{V}_2\beta\gamma}\alpha = \mathrm{C}_{\mathrm{V}_2(\mathrm{V}_1\zeta\omega)\gamma}\zeta$$

by (1) above.

It is useful to modify the cyclic rule for a self-conjugate ϖ_x, namely, ϖ_0, ϖ_a, ϖ_b, or ϖ_c. Thus, using ϖ_a as an example, we have $\Sigma \mathrm{V}_1\alpha\varpi_a \mathrm{V}_2\beta\gamma = 0$, or operating by $\mathrm{V}_0\delta(\;)$, where δ is a fourth vector,

$$\mathrm{V}_0 \,.\, \mathrm{V}_2\alpha\beta\varpi_a \mathrm{V}_2\gamma\delta + \mathrm{V}_0 \,.\, \mathrm{V}_2\alpha\gamma\varpi_a \mathrm{V}_2\delta\beta + \mathrm{V}_0 \,.\, \mathrm{V}_2\alpha\delta\varpi_a \mathrm{V}_2\beta\gamma = 0.$$

The expression on the left is combinatorial in $\alpha, \beta, \gamma, \delta$. Taking $\alpha, \beta, \gamma, \delta$ to be any four of the primitive vectors $\iota_1, \iota_2, \ldots \iota_n$, say, $\iota_1, \iota_2, \iota_3, \iota_4$, we have successively the following three statements:—

$$\begin{gathered}\Sigma\left[\iota_1{}^{-1}\iota_2{}^{-1}\iota_3{}^{-1}\iota_4{}^{-1}\Sigma \mathrm{V}_0(\iota_1\iota_2)\varpi_a(\iota_3\iota_4)\right] = 0,\\ \mathrm{V}_4\zeta_1\zeta_2\zeta_3\zeta_4 \,.\, \mathrm{V}_0(\mathrm{V}_2\zeta_1\zeta_2)\varpi_a(\mathrm{V}_2\zeta_3\zeta_4) = 0,\\ \mathrm{V}_4(\mathrm{V}_2\zeta_1\zeta_2)\varpi_a(\mathrm{V}_2\zeta_1\zeta_2) = 0, \qquad (31)\end{gathered}$$

as mentioned above between equations (12) and (13).

From equation (27) onwards we have been at no pains to put our results in invariant form. Nevertheless (27), (30), and (31) are already in invariant form, and (29) yields

$$C_\omega \gamma = CV_1 \omega\eta\gamma - C_{V_2(V_3\zeta\omega)\eta\gamma}\eta^{-1}\zeta. \tag{29a}$$

§ 17. *Comparison with Theory of Tensors.*—Let $\phi(q_1\, q_2, \ldots q_a)$ be any contravariant or covariant function linear in each of the multenions $q_1, q_2, \ldots q_a$, each one of which is also contravariant or covariant, not necessarily all of one type. Thus, for instance, $\Omega_{V_2\alpha\beta}\eta^{-1}\gamma^\times$ is such a covariant function of $\alpha, \beta, \gamma^\times$, two contravariant vectors, and one covariant vector. In the pages above we have defined various increments of q due to changing ρ to $\rho+\alpha$, where α is an arbitrary infinitesimal contravariant vector

$$(d_\alpha{}^{\mathrm{T}},\ d_\alpha{}^{\mathrm{I}},\ d_\alpha{}^{\mathrm{O}},\ d_\alpha{}^{\mathrm{A}})\, q$$

and we might add, when q is of power c, *the absolute increment proper*, also of power c, $d_\alpha{}^{\mathrm{AP}}q = d_\alpha{}^{\mathrm{B}}q = (d_\alpha{}^{\mathrm{A}} + cV_0\alpha\lambda)\, q$. The increments of ϕ of these types are all formed on the same principle. Thus take $d_\alpha{}^{\mathrm{A}}$ as the type and use the abbreviated notation $d_\alpha{}^{\mathrm{A}}q = q_\alpha$. Then the increment ϕ_α of ϕ is given by

$$\phi_\alpha(q_1, q_2, \ldots q_a)$$

$$= [\phi(q_1, q_2, \ldots q_a)]_\alpha - \sum_{c=1}^{a} \phi(q_1, q_2, \ldots q_{c\alpha} \ldots q_a)$$

where $q_{c\alpha}$ occupies the position of q_c in $\phi(q_1, q_2, \ldots, q_a)$. Once having obtained our increment, we may cease to regard α as infinitesimal, so that $d_\alpha{}^{\mathrm{A}}$ becomes, not so much an *increment* as a *rate of increase* multiplied by the magnitude of α, though above it has always been called an increment.

All this is on the model of the Theory of Tensors. In that theory, however, the only linear form $\phi(q_1, q_2, \ldots, q_a)$ recognised is the scalar form $\phi(\alpha_1, \alpha_2, \ldots, \alpha_a)$ in several vectors, some of which are covariant and the rest contravariant. The present writer marvels at the great results obtained by expounders of the Theory of Tensors, since the basis is such a limited concept.

The precise translation from our presentation to that of the Theory of Tensors and *vice versâ* is to a certain extent arbitrary, on account of vagueness, especially in the matter of sign, that writers on the Theory of Tensors seem to permit themselves. The following will be found to form one consistent system. Let A^a, B^b be two contravariant vectors, α and β, and let X_{ab} be a covariant tensor of the second rank, with the same intrinsic meaning as ξ, a covariant vector linity. A_aB_b seems to be taken to mean the same as B_bA_a in the Theory of Tensors, but I cannot see how this can be consistently done, and in the system here given A_aB_b and B_bA_a, when interpreted as vector linities, must have different meanings.

With the basis as given in § 11 above, we have as follows. Let

$$\alpha^\times = \eta\alpha, \qquad \beta^\times = \eta\beta, \tag{1}$$

$$\xi^\times = \eta^{-1}\xi\eta^{-1}, \qquad \mathrm{X} = \xi\eta^{-1}, \qquad \mathrm{X}^\times = \eta^{-1}\xi. \tag{2}$$

Then

$$\nabla = -\sum_{a=1}^{n} \iota_a{}^{-1}\frac{\partial}{\partial x_a}, \qquad \frac{\partial}{\partial x_a} = -\mathrm{V}_0\iota_a\nabla, \tag{3}$$

$$\mathrm{A}^a = \mathrm{V}_0\iota_a{}^{-1}\alpha, \qquad \mathrm{A}_a = \mathrm{V}_0\iota_a\alpha^\times, \tag{4}$$

$$g_{ab} = \mathrm{V}_0\iota_a\eta\iota_b, \qquad g^{ab} = \mathrm{V}_0\iota_a{}^{-1}\eta^{-1}\iota_b{}^{-1}, \qquad g_a{}^b = \mathrm{V}_0\iota_a\iota_b{}^{-1}, \tag{5}$$

$$[bc, a] = \mathrm{V}_0\iota_a\overline{\Gamma}_{\iota_b}\iota_c, \qquad \{bc, a\} = \mathrm{V}_0\iota_a{}^{-1}\overline{\mathrm{E}}_{\iota_b}\iota_c \tag{6}$$

$$\mathrm{X}_{ab} = \mathrm{V}_0\iota_a\xi\iota_b, \qquad \mathrm{X}_a{}^b = \mathrm{V}_0\iota_a\mathrm{X}\iota_b{}^{-1}, \qquad \mathrm{X}^{ab} = \mathrm{V}_0\iota_a{}^{-1}\xi^\times\iota_b{}^{-1}. \tag{7}$$

The equation $\mathrm{X}_{ab} = \mathrm{A}_a\mathrm{B}_b$ consistently with the above and the related equations may be interpreted to mean as follows

$$\mathrm{X}_{ab} = \mathrm{A}_a\mathrm{B}_b, \qquad \xi = \eta\alpha\mathrm{V}_0(\ )\eta\beta = \alpha^\times\mathrm{V}_0(\ )\beta^\times, \tag{8}$$

$$\mathrm{X}_a{}^b = \mathrm{A}_a\mathrm{B}^b, \qquad \mathrm{X} = \eta\alpha\mathrm{V}_0(\ )\beta = \alpha^\times\mathrm{V}_0(\ )\beta. \tag{9}$$

$$\mathrm{X}^{ab} = \mathrm{A}^a\mathrm{B}^b, \qquad \xi^\times = \alpha\mathrm{V}_0(\ )\beta. \tag{10}$$

Now (8) and (10) definitely tell us what $\mathrm{B}_b\mathrm{A}_a$ and $\mathrm{B}^b\mathrm{A}^a$ must mean; instead of representing ξ and $\xi^\times$ respectively, they represent ξ' and $\xi^{\times\prime}$ and $\mathrm{B}^b\mathrm{A}_a$ is similarly naturally interpreted as $\eta\beta\mathrm{V}_0()\alpha$, that is as $\eta\mathrm{X}'\eta^{-1}$.

Lastly we have

$$(\zeta, \zeta) = -\sum_{a=1}^{n}(\iota_a, \iota_a{}^{-1}). \tag{11}$$

It is with these interpretations that equation (25), § 16, was stated to be equivalent to Eddington's (26·3).

I may add here that Eddington's equation (23) for $\mathrm{B}_{abc}{}^h$ gives

$$\mathrm{B}_{abc}{}^h = -\mathrm{V}_0\iota_h{}^{-1}\overline{\mathrm{C}}_{\mathrm{V}_2\iota_b\iota_c}\iota_a \tag{12}$$

where $\overline{\mathrm{C}}_\omega$ means $\mathrm{C}_{\omega a}$, that is the part of C_ω obtained by putting $\lambda = 0$.

It will be seen that I interpret the mixed tensor to mean a comixt, not contramixt, vector linity.

18. *Prof. Eddington's recent work.*[*] [Re-written September, 1921.]

Suppose ABC is a triangle whose sides AB, BC, CA are geodesics. Let the point O of an infinitesimal lattice-work pass round the triangle, starting and ending at A, and in each stage of the journey let the lattice-work suffer "parallel displacement" only. When the circuit has been completed, then:—according to Euclid and Galileo the lattice-work will be found occupying its initial position exactly; according to Riemann it will be found turned through an angle in a plane containing O; according to Weyl it will be turned and changed in size, but not in shape; according to Eddington it will be found turned, changed in size and also in shape. If Prof. Eddington meant no more than this, his manifold, as will be shown below, would be mathematically equivalent to Weyl's, but he does mean more. In his primary manifold *there is no criterion for shape at all,* none to distinguish a square from any other parallelogram, or to distinguish a cube from any other parallelepiped. In Riemann's manifold the fundamental idea is that of the invariant quadratic form (squared interval), and "parallel displacement" is a mathematically *derived* idea. In Eddington's manifold the *fundamental* idea is that of parallel displacement, and an invariant quadratic form is a mathematically derived idea. When $n = 4$, Riemann's, Weyl's and Eddington's foundations require ten, thirteen (nine *ratios* of coefficients of quadratic form and four coefficients of a linear form) and forty scalars at a point to be given, respectively.

---

[*] 'Roy. Soc. Proc.,' A, vol. 99, p. 104 (1921).

Eddington then starts with $\mathrm{E}_a\beta (= \mathrm{E}_\beta\alpha, - \mathrm{E}_a\beta$ meaning increment of β due to the parallel displacement α; α, β being contravariant vectors) but without any fundamental η specifying the intrinsic invariant $\mathrm{V}_0 d\rho\eta d\rho$. *All our previous work and results, which are independent of* η, *or can be made so, continue to be valid in Eddington's manifold.* On looking back I only see one case where our old reasoning ceases to hold in the new circumstances. This is equation (4), § 15, which was proved by aid of η. But the equation is still true because the condition that τ, a contravariant vector, suffers the parallel displacement β is

$$-\mathrm{V}_0\beta\nabla\,.\,\tau = -\mathrm{E}_\beta\tau,$$

and this is an intrinsic condition independent of choice of coordinates. Infinitesimally change the coordinates as in § 15 $[\delta\tau = \chi_0\tau,\ \ \delta\beta = \chi_0\beta,$ $\delta(\mathrm{V}_0\beta\nabla) = 0]$. Thus

$$-\mathrm{V}_0\beta\nabla\,.\,(\chi_0\tau) = -\delta(\mathrm{E}_\beta\tau);$$

and here the left transforms to

$$-\chi_0(\mathrm{V}_0\beta\nabla\,.\,\tau) - \mathrm{V}_0\beta\nabla_9\,.\,\chi_{09}\tau = -\chi_0\mathrm{E}_\beta\tau - \mathrm{V}_0\beta\nabla_9\,.\,\chi_{09}\tau$$

so that

$$\delta(\mathrm{E}_\beta\tau) = \chi_0\mathrm{E}_\beta\tau + \mathrm{V}_0\beta\nabla_9\,.\,\chi_{09}\tau.$$

This is equation (4) § 15 with τ in place of γ. We may then make the very important use of that equation which we made before, namely, if τ is *any* contravariant vector,

$$d_a{}^{\mathrm{A}}\tau = -\mathrm{V}_0\alpha\nabla\,.\,\tau + \mathrm{E}_a\tau$$

is contravariant and may therefore be appropriately called the *absolute* increment of τ. [Prof. Eddington does not show in his paper that absolute differentiation thus holds good in the "in-" sense in his manifold.]

Our chief disability from absence of a fundamental η seems to be that there is now found no place for the ideas associated with normal and incident components (and expressions) and with rotations. $\mathrm{V}_{a+b}uv$ and $\mathrm{V}_{a-b}uv$ have their invariantive equivalents, but apparently $\mathrm{V}_{a+b-2c}uv$ in general has not. We can still speak of α and $\beta^{\times}$ being normal when $\mathrm{V}_0\alpha\beta^{\times} = 0$ and this leaves us with (what we met with in § 10 before η was introduced) the ideas of normal and incident components with reference not to a single τ but with reference to τ_c, some one of n given independent contravariant vectors $\tau_1, \tau_2, \ldots \tau_n$. In a word the normal reciprocals of $\tau_1, \tau_2, \ldots \tau_n$ continue to have invariantive meanings.

For present purposes the contraction we selected in § 16 to pass from C_ω to Ω was unhappily chosen as it depended on η. We will here, therefore, change the contraction to another. The previous definition of $\Omega\beta$ was that it was obtained by setting α, γ equal to $\zeta, \eta^{-1}\zeta$ in $\eta\mathrm{C}_{\mathrm{V}_2\alpha\beta}\gamma$; our new meaning of $\Omega'\alpha$ is obtained by setting $\beta, \gamma^{\times}$ equal to ζ, ζ in $\mathrm{C}_{\mathrm{V}_2\alpha\beta}'\gamma^{\times}$. Thus

$$\left.\begin{array}{ll}\text{Previous} & \Omega\beta = \eta\mathrm{C}_{\mathrm{V}_2\zeta\beta}\eta^{-1}\zeta \\ \text{Present} & \Omega'\alpha = \mathrm{C}_{\mathrm{V}_2\alpha\zeta}'\zeta\end{array}\right\}. \tag{1}$$

In Weyl's manifold the change in meaning is slight. The old and new self-conjugate parts are the same, but whereas the previous skew part of $\Omega\beta$ was $\frac{1}{4}(n-4)\mathrm{V}_1\omega^{\times}\beta$, the present is $-\frac{1}{4}n\mathrm{V}_1\omega^{\times}\beta$. This simplicity of connections between old and new meanings may be proved from the identity [equation (8), § 16].

$$\eta\mathrm{C}_{\mathrm{V}_2\alpha\beta}\gamma + \mathrm{C}_{\mathrm{V}_2\alpha\beta}'\eta\gamma = -\mathrm{V}_0\omega^{\times}\mathrm{V}_2\alpha\beta\,.\,\eta\gamma,$$

by setting α, γ equal to $\zeta, \eta^{-1}\zeta$. The identity is the one which expresses that our lattice-work, after its journey round the circuit ABC, is changed in size but not in shape.

Our equation for C_ω (4) of § 16 still holds while η is unassigned, and from it

$$\mathrm{C}_\omega'\gamma^{\times} = -\mathrm{E}_{9\,.\,\mathrm{V}_1\omega\nabla_9}'\gamma^{\times} + (-\mathrm{E}_\beta'\mathrm{E}_\alpha' + \mathrm{E}_\alpha'\mathrm{E}_\beta)\gamma^{\times}. \tag{2}$$

Putting here $\mathrm{V}_1\omega\nabla = \alpha\mathrm{V}_0\beta\nabla - \beta\mathrm{V}_0\alpha\nabla$ and using (1) we get

$$\Omega'\alpha = \mathrm{E}_{9.\alpha}'\nabla_9 + \mathrm{V}_0\alpha\nabla\,.\,\mathrm{E}_\zeta'\zeta - \mathrm{E}_\zeta\mathrm{E}_\alpha'\zeta + \mathrm{E}_\alpha'\mathrm{E}_\zeta'\zeta. \tag{3}$$

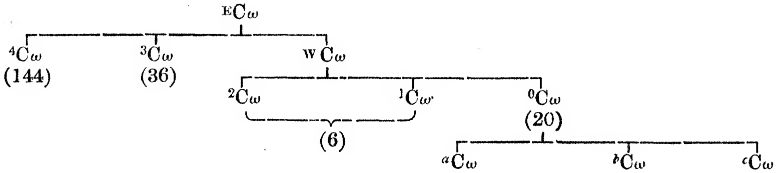
The first, third and fourth terms on the right are all self-conjugate because $\mathrm{E}_\alpha\beta = \mathrm{E}_\beta\alpha$. Hence the skew part of $\Omega\alpha$ is $-\frac{1}{2}\mathrm{V}_1\alpha\mathrm{V}_2\nabla\mathrm{E}_\zeta'\zeta$.

By its original meaning $\mathrm{C}_\omega\gamma$ is the difference of two truly contravariant, or as Eddington says in-contravariant (would not *preter*-contravariant, be better ?) vectors at the same point. Hence C_ω, C_ω', Ω and $\mathrm{V}_2\nabla\mathrm{E}_\zeta'\zeta$ all have the "in-" property and $\frac{1}{2}(\Omega + \Omega')$ is fitted for serving as the fundamental η of a Riemann manifold. Prof. Eddington, therefore, identifies

$\frac{1}{2}cV_0 d\rho(\Omega + \Omega')d\rho$ with Einstein's quadratic form, the squared interval; c being an absolute constant. I do not propose to follow him in this respect, but to show instead, that as soon as it is recognized that "lengths" can be compared (even in Weyl's limited sense that equality of lengths can be postulated *at* a point only), then Eddington's theory is mathematically equivalent to Weyl's; and instead of merely one simple quadratic form there are several available to identify with Einstein's.

It is now necessary to distinguish between Weyl's and Eddington's meanings of E_α, C_ω, Ω and related symbols. For Eddington's meanings we shall use $\mathbf{E}_\alpha$, ${}^{E}C_\omega$, ${}^{E}\Omega$. For Weyl's we shall use E_α, ${}^{W}C_\omega$, ${}^{W}\Omega$.

The following three modifications of notations occurring in §§ 14 to 16 above are desirable: (1) The letter G will be used in place of the letter F, because F is wanted below for another purpose; (2) the "contraction" to pass from C_ω to Ω (both of power zero), will be altered in meaning as already described; (3) the symbols 2, 1, 0, a, b, c, used as suffixes to C_ω, Ω_ω, C, Ω, ϖ, G (former F), will be shifted from the right-hand bottom corner of the letter affected to the left-hand top corner. The following scheme will probably be helpful:—



[In each of these four lines there is one symbol equivalent to a tensor of Eddington's, or to a tensor of Weyl's. ${}^{E}C_\omega$ is equivalent to Eddington's ${}^{*}B_{abc}{}^{h}$, ${}^{W}C_\omega$ to Weyl's $F_{abc}{}^{h}$, ${}^{0}C_\omega$ to Weyl's ${}^{*}F_{abc}{}^{h}$, and ${}^{a}C_\omega$ to both Eddington's $B_{abc}{}^{h}$ and Weyl's $R_{abc}{}^{h}$.] In the scheme the E of ${}^{E}C_\omega$ implies "in Eddington's sense"; the W of ${}^{W}C_\omega$ implies "in Weyl's sense"; further, the scheme implies the following three defining equations:—

$$
\begin{aligned}
{}^{E}C_\omega &= {}^{4}C_\omega + {}^{3}C_\omega + {}^{W}C_\omega, \\
{}^{W}C_\omega &= {}^{2}C_\omega + {}^{1}C_\omega + {}^{0}C_\omega, \\
{}^{0}C_\omega &= {}^{a}C_\omega + {}^{b}C_\omega + {}^{c}C_\omega.
\end{aligned}
$$

We may write x for any one of the ten symbols E, W, 4, 3, 2, 1, 0, a, b, c. For seven out of the ten meanings of ${}^{x}C_\omega$, namely, for all except ${}^{a}C_\omega$, ${}^{b}C_\omega$, ${}^{c}C_\omega$, it is true that ${}^{x}C_\omega$ is of zero power as meant by Weyl. The three ${}^{a}C_\omega$, ${}^{b}C_\omega$, ${}^{c}C_\omega$, are invariantive for change of co-ordinates but not for change of gauge. The numbers in brackets (144), etc., imply that ${}^{E}C_\omega$ involves, when $n = 4$, $206 (= 144 + 36 + 6 + 20)$, independent scalars, that ${}^{W}C_\omega$ involves $26 (= 6 + 20)$, independent scalars, and so on. [It is explained below that the statement that ${}^3\mathrm{C}_\omega$ involves 36 scalars, merely means that ${}^3\mathrm{C}_\omega$ is fully known when the 36 scalars of $\Upsilon_\alpha\beta$ below are known; and, again, ${}^4\mathrm{C}_\omega$ is fully known when the 144 first derivatives of the 36 scalars are known. In a purely algebraic sense ${}^4\mathrm{C}_\omega$ cannot involve more than 96 scalars.]

In what now follows Eddington's symbols

$$\Gamma_{ab}^{c},\quad {}^*\mathrm{B}_{abc}^{h},\quad {}^*\mathrm{G}_{ab},\quad \mathrm{R}_{ab},\quad \mathrm{F}_{ab},\quad \mathrm{K}_{ab,\,c},\quad \mathrm{S}_{ab,\,c},\quad 2\kappa_a,$$

are the equivalents of our

$$\mathbf{E}_a,\quad {}^{\mathrm{E}}\mathrm{C}_\omega,\quad {}^{\mathrm{E}}\Omega,\quad \tfrac{1}{2}({}^{\mathrm{E}}\Omega+{}^{\mathrm{E}}\Omega'),\quad \tfrac{1}{2}({}^{\mathrm{E}}\Omega-{}^{\mathrm{E}}\Omega'),\quad \Psi_\beta\gamma,\quad \Lambda_\beta\gamma,\quad \mathbf{E}_\zeta{}'\zeta,$$

respectively.

In our interpretations, Eddington's "increment due to parallel displacement" has to be distinguished from our (for Weyl's manifold), "increment due to translation." We use $d_\alpha{}^{\mathrm{PD}}$ and $d_\alpha{}^{\mathrm{T}}$ for these phrases. As with (5) of § 14 above, we have

$$d_\alpha{}^{\mathrm{PD}}\mathrm{V}_0\beta\eta\gamma = -\mathrm{V}_0\alpha\nabla_9\,.\,\mathrm{V}_0\beta\eta_9\gamma-\mathrm{V}_0\mathbf{E}_\alpha\beta\eta\gamma-\mathrm{V}_0\beta\eta\mathbf{E}_\alpha\gamma.$$

Prof. Eddington remarks that as this is the difference of two invariants it must be invariant, and therefore must be of the form $2\mathrm{V}_0\alpha\Psi_\beta\gamma$ (a more general form than ours, after Weyl, $-\mathrm{V}_0\beta\eta\gamma\,.\,\mathrm{V}_0\lambda\alpha$ in equation (4) of § 14), where $\Psi_\beta\gamma$ is a covariant vector linear and symmetrical in β and γ (contravariant vector dummies), given by

$$2\Psi_\beta\gamma = -\nabla_9\mathrm{V}_0\beta\eta_9\gamma-\mathbf{E}_\beta{}'\eta\gamma-\mathbf{E}_\gamma{}'\eta\beta. \tag{4}$$

$\Psi_\beta\gamma$ can easily be expressed uniquely as the sum of two parts, a contractile part involving n scalars and a non-contractile part involving $\frac{1}{2}(n+2)n(n-1)$ scalars. The contractile part is $-n^{-1}\Psi_\zeta\eta^{-1}\zeta\,.\,\mathrm{V}_0\gamma\eta\beta$, and is so named because it is the sole part of $\Psi_\beta\gamma$ which is responsible for the contraction $\Psi_\zeta\eta^{-1}\zeta$, a vector. Thus

$$2\Psi_\beta\gamma = -\lambda\mathrm{V}_0\beta\eta\gamma+2\Upsilon_\beta\gamma \tag{5}$$

where λ is a covariant vector (namely, $2n^{-1}\Psi_\zeta\eta^{-1}\zeta$), and $\Upsilon_\beta\gamma$ is a covariant vector, linear and symmetrical in β and γ, and satisfying the vector equation

$$\Upsilon_\zeta\eta^{-1}\zeta = 0. \tag{6}$$

[Digression.—If $Z_\alpha\beta$ is contravariant (and similar remarks hold when it is covariant), and does not satisfy the equation of symmetry $Z_\alpha\beta = Z_\beta\alpha$, there are three kinds of contraction (derived from the scalar form $\mathrm{V}_0\gamma^\times Z_\alpha\beta$). The corresponding contractile parts of $Z_\alpha\beta$ are

$$-\eta^{-1}\zeta_1\,.\,\mathrm{V}_0\zeta_2Z_{\zeta_1}\zeta_2\,.\,\mathrm{V}_0\alpha\eta\beta,\quad \eta^{-1}Z_\zeta{}'\zeta\,.\,\mathrm{V}_0\alpha\eta\beta,\quad -\eta^{-1}Z_\zeta\eta^{-1}\zeta\,.\,\mathrm{V}_0\alpha\eta\beta,$$

the three contractions themselves being

$$-\zeta_1\,.\,\mathrm{V}_0\zeta_2Z_{\zeta_1}\zeta_2,\qquad Z_\zeta{}'\zeta,\qquad Z_\zeta\eta^{-1}\zeta.$$

The first and second of these contractions are equal when $Z_\alpha\beta = Z_\beta\alpha$.]

The λ appearing in equation (5) is the λ we met with in Weyl's theory. If we examine what change occurs in $\Psi_\beta\gamma$, or better in $\eta^{-1}\Psi_\beta\gamma$, due to change of gauge, we at once find that in $\eta^{-1}\Psi_\beta\gamma$ the whole change falls on the λ part, no change at all occurring in $\eta^{-1}\Upsilon_\beta\gamma$; that is $\eta^{-1}\Upsilon_\beta\gamma$ is of zero power.

Calculating from (4) the value of

$$V_0\alpha\Lambda_\beta\gamma = V_0\alpha\Psi_\beta\gamma - V_0\beta\Psi_\gamma\alpha - V_0\gamma\Psi_\alpha\beta$$

we at once find that

$$\Lambda_\beta\gamma = (\Psi_\beta\gamma - \Psi_\beta'\gamma) - \Psi_\gamma'\beta \tag{7}$$

and

$$\mathbf{E}_\beta\gamma = E_\beta\gamma + Y_\beta\gamma \tag{8}$$

where

$$Y_\beta\gamma = \eta^{-1}(\Upsilon_\beta\gamma - \Upsilon_\beta'\gamma - \Upsilon_\gamma'\beta) \tag{9}$$

and $E_\beta\gamma$ is exactly as in § 14. On the right of (7) the two bracketed terms furnish the skew part of Λ_β and the third term the self-conjugate part. It will be easily seen that

$$Y_\zeta'\zeta = -\Upsilon_\zeta\eta^{-1}\zeta = 0. \tag{10}$$

From (9) $Y_\beta\gamma$ is of zero power. It follows that it is just as legitimate to name $-E_\beta\gamma$ the increment of γ due to the translation $\beta\,(d_\beta{}^{\mathrm{T}}\gamma)$ as it is to name $-\mathbf{E}_\beta\gamma$ the increment of γ due to the parallel displacement $\beta\,(d_\beta{}^{\mathrm{PD}}\gamma)$. In other words Eddington's manifold is simply Weyl's in which there is supposed to exist a given $Y_\beta\gamma$ of zero power and satisfying the equation $Y_\zeta'\zeta = 0$.

With our present notation [see (4) of § 16]

$${}^{\mathrm{E}}C_\omega\gamma = d_\omega{}^{\mathrm{PD}}\gamma = -\mathbf{E}_{9.V_1\omega\nabla_9}\gamma + (-\mathbf{E}_\alpha\mathbf{E}_\beta + \mathbf{E}_\beta\mathbf{E}_\alpha)\gamma. \tag{11}$$

Putting $\mathbf{E}_\alpha = E_\alpha + Y_\alpha$ we get a part involving E_α, E_β only. This is, of course, our former

$${}^{\mathrm{W}}C_\omega\gamma = d_\omega{}^{\mathrm{T}}\gamma = -E_{9.V_1\omega\nabla_9}\gamma + (-E_\alpha E_\beta + E_\beta E_\alpha)\gamma. \tag{12}$$

Next we get a second part

$${}^{3}C_\omega\gamma = (-Y_\alpha Y_\beta + Y_\beta Y_\alpha)\gamma \tag{13}$$

and a third part which may be written

$${}^{4}C_\omega\gamma = -(Y_\alpha\gamma)_\beta + (Y_\beta\gamma)_\alpha \tag{14}$$

where $(\quad)_\beta$ and $(\quad)_\alpha$ denote absolute increments in the sense developed in § 15 above for Weyl's manifold. (14) follows from the beginning of § 17 from which is deducible

$$(Y_\alpha\gamma)_\beta = -Y_{9\alpha}\gamma \,.\, V_0\beta\nabla_9 + E_\beta Y_\alpha\gamma - Y_\alpha E_\beta\gamma - Y_\gamma E_\beta\alpha.$$

With $n = 4$ (13) involves the thirty-six scalars of $Y_\alpha\beta$ and (14) involves their 144 first derivatives (∇). The full meaning of our scheme for ${}^{\mathrm{E}}C_\omega$, given above, will now be evident.

There are numerous simple quadratic forms available to identify, in Prof. Eddington's manner, with the square of Einstein's interval. Not only may we contract (12) and select its self-conjugate part ${}^0\Omega$, but we may similarly contract each of (13) and (14) in two different ways and this by no means exhausts our available simple forms. Besides (13) and (14) we may similarly use $(Y_\alpha Y_\beta + Y_\beta Y_\alpha)\gamma$ and $(Y_\alpha \gamma)_\beta + (Y_\beta \gamma)_\alpha$ where yet another kind of contraction is available (put $\alpha, \beta = \zeta, \eta^{-1}\zeta$).

Returning to the use we made of (3) above, we see that the skew part of ${}^E\Omega\alpha$ is $-\frac{1}{2}V_1\alpha V_2\nabla\mathbf{E}_\zeta'\zeta$. Since $Y_\zeta'\zeta = 0$,

$$\mathbf{E}_\zeta'\zeta = E_\zeta'\zeta = \nabla \log k - \tfrac{1}{2}n\lambda, \tag{15}$$

by (18) of § 15 above. Hence Prof. Eddington's identification of $V_2\nabla\mathbf{E}_\zeta'\zeta$ in the electromagnetic field is virtually the same as Weyl's identification of $V_2\nabla\lambda$. Also note that ${}^E\Omega - {}^W\Omega\,(= {}^3\Omega + {}^4\Omega)$ is self-conjugate.

One consequence of Eddington's suggestions is that we may quite likely desire to identify Einstein's squared interval with something very different from our fundamental form $V_0 d\rho\eta d\rho$. Let Einstein's squared interval be $V_0 d\rho\theta d\rho$. We may wish to put $\theta = \frac{1}{2}({}^E\Omega + {}^E\Omega')$ or $\theta = {}^0\Omega = \frac{1}{2}({}^W\Omega + {}^W\Omega')$ or $\theta = \frac{1}{2}({}^3\Omega + {}^3\Omega')$, the last making θ depend solely on $Y_\alpha\beta$. With any such identification we require symbols dependent on θ, as are k, ${}^W\Omega$, etc., on η. We shall, therefore, henceforth use

$$\theta,\ l,\ F_\beta,\ \Theta_\beta,\ C_\omega,\ \Omega,\ \varpi,\ G,$$

in the θ manifold (Riemann manifold) in the same way as above

$$\eta,\ k,\ E_\beta,\ \Gamma_\beta,\ {}^WC_\omega,\ {}^W\Omega,\ {}^W\varpi,\ {}^WG,$$

have been used in the η manifold (Weyl manifold). The relations between θ, l, F_β, etc. are precisely the same as those between

$$\eta,\ k,\ {}^aE_\beta,\ {}^a\Gamma_\beta,\ {}^aC_\omega,\ {}^a\Omega,\ {}^a\varpi,\ {}^aG,$$

(${}^aE_\beta$, ${}^a\Gamma_\beta$ here replacing our earlier $\overline{E}_\beta$, $\overline{\Gamma}_\beta$).

If θ is identified in some such manner as just suggested the determinant $|\theta|$ may become zero. $|\theta| = 0$ and $|\theta^{-1}| = 0$ represent loci. Do the loci if really existent, constitute the boundaries of electrons and atomic nuclei? On crossing the boundary do we pass to a region with four real dimensions or to a region of two real and two imaginary dimensions (two-dimensional time and two-dimensional space)? Is it on the other hand meaningless to speak about crossing the boundary, much as to an inhabitant of hyperbolic space it would be meaningless to speak of passing across his space's boundary, the absolute?